Math 629, Spring 2020 (Roos) – Homework 10.

Optional homework sheet – not graded.

(*) asterisk denotes problems that may be more challenging.

1. Let $\mathcal{D} = \{[2^k\ell, 2^k(\ell+1)) : k, \ell \in \mathbb{Z}\}$ the collection of dyadic intervals. As seen in class, define $h_I = |I|^{-1/2}(\mathbf{1}_{I_l} - \mathbf{1}_{I_r})$ for every $I \in \mathcal{D}$, where $I_l, I_r \in \mathcal{D}$ are the left, resp. right halves of I.

(a) Show that $\{h_I : I \in \mathcal{D}\}$ is an orthonormal system on $L^2(\mathbb{R})$.

(b*) Denote $\mathbb{E}_n f = \sum_{I\in\mathcal{D},\, |I|>2^n} \langle f, h_I\rangle h_I$ for compactly supported, locally integrable f (note the sum is finite). Show that

$$\mathbb{E}_n f = \sum_{I\in\mathcal{D},\, |I|=2^n} \Big(|I|^{-1}\int_I f\Big)\mathbf{1}_I.$$

(c) Show that $\mathbb{E}_n$ extends to a bounded linear operator on $L^2(\mathbb{R})$.

(d*) Show that $(h_I)_{I\in\mathcal{D}}$ is a complete orthonormal system on $L^2(\mathbb{R})$.

2. Let $\mathcal{H}$ be a Hilbert space and $K \subset \mathcal{H}$ a closed convex set that is not empty. Let $f \in \mathcal{H}$.

(a) Show that there exists a unique $g_* \in K$ so that

$$\|f - g_*\| = \inf_{g\in K} \|f - g\|.$$

(We proved this in class if K is a closed linear subspace.)

(b*) Is it necessarily true that $\langle f - g_*, g\rangle = 0$ for all $g \in K$?

3. Recall that a normed vector space is called separable if it contains a countable dense subset.

(a) Prove that $L^p(\mathbb{R}^d)$ is separable for every $p \in [1, \infty)$.

(b) Prove that $L^\infty(\mathbb{R}^d)$ is not separable.

(c*) Prove that if a Hilbert space is separable, then it contains a complete orthonormal system. *Hint:* Use Gram-Schmidt orthogonalization.

4. In this exercise we consider the measure space $(0, \infty)$, equipped with the Lebesgue measure. Define the linear operator

$$Hf(x) = \frac{1}{x}\int_0^x f(t)dt, \quad x \in (0, \infty),$$

acting on functions $f : (0, \infty) \to \mathbb{C}$.

(a) Show that if $f \in L^1$, then not necessarily $Hf \in L^1$.

(b*) Show that H extends to a bounded operator $L^p \to L^p$ for every $p \in (1, \infty]$. *In this exercise you are not allowed to use the Hardy–Littlewood maximal function.*