Math 629, Spring 2020 (Roos) – Homework 11.
Optional homework sheet – not graded.
(*) asterisk denotes problems that may be more challenging.

1. Let $\mathbb{E}_k f = \sum_{I\in\mathcal{D},|I|=2^k}(|I|^{-1}\int_I f)\mathbf{1}_I$ denote the dyadic martingale operators from Homework 10. Prove that for all $k,\ell\in\mathbb{Z}$ and non-negative measurable functions f,

$$\int_0^1 f(x)\mathbb{E}_k f(x)\mathbb{E}_\ell f(x)dx \ge c\Big(\int_0^1 f(x)dx\Big)^3,$$

where $c\in(0,\infty)$ is a constant independent of f,k,ℓ.

2. Let $\mathcal{H}$ be a Hilbert space over $\mathbb{K}\in\{\mathbb{R},\mathbb{C}\}$.
(i) Let $\ell:\mathcal{H}\to\mathbb{K}$ be a bounded linear map. Then there exists a unique $g\in\mathcal{H}$ such that $\ell(f,g)=\langle f,g\rangle$ for all $f\in\mathcal{H}$. *Hint:* Consider the closed subspace $\ker\ell$ and use orthogonal projection.
(ii) Let $T:\mathcal{H}\to\mathcal{H}$ be a bounded linear map. Show that there exists a unique bounded linear map $T^*:\mathcal{H}\to\mathcal{H}$ (called the *adjoint* of T) such that $\langle Tf,g\rangle=\langle f,T^*g\rangle$ for all $f,g\in\mathcal{H}$ and $\|T\|=\|T^*\|$. *Hint:* Use (i).

3. Let $\mathcal{H}$ be a Hilbert space and $\{e_n : n=0,1,\dots\}\subset\mathcal{H}$ a complete orthonormal system. Let $\{\lambda_n : n=0,1,\dots\}\subset\mathbb{K}$ be a given bounded sequence of scalars. Define a linear map $T:\mathcal{H}\to\mathcal{H}$ by $Tf=\sum_{n=0}^\infty\lambda_n\langle f,e_n\rangle e_n$.
(i) Prove that Tf is well-defined as an element of $\mathcal{H}$ (that is, verify that the infinite series defining Tf converges in $\mathcal{H}$).
(ii) Prove that T is a bounded linear operator with operator norm $\|T\|=\sup_{f\in\mathcal{H},f\neq0}\|Tf\|/\|f\|$ given by $\|T\|=\sup_n|\lambda_n|$.
(iii) Show that T is an orthogonal projection if and only if $\lambda_n\in\{0,1\}$ for all $n=0,1,\dots$.
(iv) Show that the adjoint of T^* is given by $T^*f=\sum_{n=0}^\infty\overline{\lambda_n}\langle f,e_n\rangle e_n$.

4. Let $X=(0,1]$ with the measure $\mu(E)=(\log 2)^{-1}\int_E\frac{1}{1+x}dx$ defined on Lebesgue measurable sets $E\subset X$.
(i) Show that the map $T:X\to X$ defined by $T(x)=\frac{1}{x}$ mod 1 is a measure-preserving transformation with respect to μ.
(ii*) Show that T is ergodic.