Math 629, Spring 2020 (Roos) – Homework 2.
Due Monday, February 17.

(Problems with an asterisk (*) are optional; problems with two asterisks (**) are optional and may be more challenging.)

1. Let X be a non-empty set, $\mathcal{A} \subset \mathcal{P}(X)$ an algebra, $\mu_0 : \mathcal{A} \to [0, \infty]$ a premeasure and $\mu_* : \mathcal{P}(X) \to [0, \infty]$ the outer measure induced by μ_0. Denote by $\mathcal{A}_\sigma$ the collection of sets that arise as countable unions of sets in $\mathcal{A}$ and $\mathcal{A}_{\sigma\delta}$ the collection of sets that arise as countable intersections of sets in $\mathcal{A}_\sigma$.

(i) Prove that for every $E \subset X$ there exists $F \in A_{\sigma\delta}$ such that $E \subset F$ and $\mu_*(E) = \mu_*(F)$.

(ii) Prove that for every Carathéodory measurable set E there exist $F \in A_{\sigma\delta}$ and $N \subset X$ with $\mu_*(N) = 0$ such that $E = F \Delta N$. (where Δ is symmetric set difference: $F \Delta N = (F - N) \cup (N - F)$)

2. Let μ_* denote the outer measure on $\mathbb{R}^n$ defined via boxes as seen in the lecture (also known as Lebesgue outer measure). Show that $E \subset \mathbb{R}^n$ is Carathéodory measurable if and only if for every $\varepsilon > 0$ there exists an open set $U \subset \mathbb{R}^n$ such that $E \subset U$ and $\mu_*(U \cap E^c) \leq \varepsilon$.

3. (i) Show that every $x \in [0, 1]$ may be written in the form $x = \sum_{k=1}^\infty a_k 3^{-k}$ with $a_k \in \{0, 1, 2\}$ for all k.

(ii) Define a set $C \subset [0, 1]$ by letting $x \in C$ if and only if x has a representation as in (i) with $a_k \in \{0, 2\}$ for all k. Show that C has Lebesgue measure zero.

4. Let $C \subset [0, 1]$ be the set from Problem 3. Define a map $f : C \to [0, 1]$ by letting $f(x) = \sum_{k=1}^\infty (a_k/2)2^{-k}$ where $x = \sum_{k=1}^\infty a_k 3^{-k}$, $a_k \in \{0, 2\}$.

(i) Show that f is well-defined (independent of the chosen representation of x and maps into $[0, 1]$).

(ii) Show that f is continuous.

(iii) Show that f is surjective: for every $y \in [0, 1]$ exists $x \in C$ so that $f(x) = y$.

(iv) Show that f can be extended to a continuous function on $[0, 1]$.

(Turn the page.)

5*. Show that there exist sets $A, B \subset \mathbb{R}$ with Lebesgue measure zero such that $A + B = \{a + b : a \in A, b \in B\}$ has positive Lebesgue measure. *Hint:* Recall the set C from Problem 3.

6.** Let $C \subset [0, 1]$ be the set from Problem 3. Show that there exists a set $D \subset C$ such that D is not a Borel set. (However, note that D must be Lebesgue measurable since it has outer measure zero.)

7.** Let $X = \mathbb{R} \times (0, \infty)$ denote the upper half-plane and let $\mathcal{T}$ be the set of all *tents* which we define as sets $T = T(x, s) \subset X$ of the form

$$T(x, s) = \{(y, t) \in X : t < s, |x - y| < s - t\}$$

for some $(x, s) \in X$. For every $E \subset X$ define an outer measure by

$$\mu_*(E) = \inf\Big\{ \sum_{i=1}^{\infty} s_i : \bigcup_{i=1}^{\infty} T(x_i, s_i) \supset E \Big\}.$$

(i) Let $E \subsetneq X$ be a non-empty set. Show that E is not Carathéodory measurable with respect to μ_*.

(ii) Show that $\mu_*(T(x, s)) = s$ for every $(x, s) \in X$.