Math 629, Spring 2020 (Roos) – Homework 3.
Due Monday, February 24.

(Problems with an asterisk (*) are optional; problems with two asterisks (**) are optional and may be more challenging.)

1. Let (X, Σ, μ) be a measure space and $(E_k)_{k=1,\dots}$ a sequence of sets in Σ. Assume that $\sum_{k=1}^{\infty} \mu(E_k) < \infty$. Define

$$E = \{x \in X \,:\, x \text{ is contained in infinitely many of the } E_k\}.$$

Show that $E \in \Sigma$ and that $\mu(E) = 0$.

2. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a function such that the function $x \mapsto f(x, y)$ is continuous for every $y \in \mathbb{R}$ and the function $y \mapsto f(x, y)$ is continuous for every $x \in \mathbb{R}$. Show that f is measurable (with respect to Lebesgue measure on $\mathbb{R}^2$).

3. Define the subset $A \subset \mathbb{R}$ as follows: $x \in A$ if and only if there exists $c > 0$ such that

$$|x - j2^{-k}| \ge c2^{-k}$$

holds for all integers j and all integers $k \ge 0$. Determine the Lebesgue outer measure of A.

4*. Let (X, Σ, μ) be a measure space. Suppose that N, N' are non-negative integers, $(a_k)_{k=1,\dots,N}$, $(a'_j)_{j=1,\dots,N'}$ real numbers and $(E_k)_{k=1,\dots,N}$, $(E'_j)_{j=1,\dots,N'}$ sets in Σ with finite μ-measure. Suppose that for every $x \in X$,

$$\sum_{k=1}^{N} a_k \mathbf{1}_{E_k}(x) = \sum_{j=1}^{N'} a'_j \mathbf{1}_{E'_j}(x).$$

Then

$$\sum_{k=1}^{N} a_k \mu(E_k) = \sum_{j=1}^{N'} a'_j \mu(E'_j).$$

(You are not allowed to integrate. Prove this directly from definitions.)

5.** Let $E \subset \mathbb{R}^d$ be a Lebesgue measurable set with $\mu(E) < \infty$. Let $f : E \to \mathbb{R}$ be a measurable function (with respect to Lebesgue measurable sets).

(i) Show that for every $\varepsilon > 0$ there exists $A_\varepsilon \subset E$ such that $\mu(E - A_\varepsilon) \le \varepsilon$ and the function $f|_{A_\varepsilon} : A_\varepsilon \to \mathbb{R}$ is continuous.

(ii) Show that A_ε can be chosen to be compact.

Note: Continuity of $f|_{A_\varepsilon}$ is (much) weaker than continuity of f on A_ε.