Math 629, Spring 2020 (Roos) – Homework 4.
Due Monday, March 2.

(Problems with an asterisk (*) are optional; problems with two asterisks (**) are optional and may be more challenging.)

1. Let $f : \mathbb{R} \to \mathbb{R}$ be non-decreasing (i.e. $f(x) \geq f(y)$ if $x \geq y$). Prove that f is measurable.

2. Construct a non-negative Lebesgue integrable function $f : \mathbb{R} \to [0, \infty)$ such that if $g : \mathbb{R} \to [0, \infty)$ is any function with $f = g$ almost everywhere, then g is unbounded on every open interval.

3. Let $0 < a < b < \infty$ and $f_n(x) = ae^{-nax} - be^{-nbx}$. Prove that

$$\sum_{n=1}^{\infty} \Big(\int_{[0,\infty]} f_n(x)dx \Big) = 0,$$

but

$$\int_{[0,\infty]} \Big(\sum_{n=1}^{\infty} f_n(x) \Big) dx = \log(b/a).$$

(Hint: The integrals are to be understood as Lebesgue integrals; but you are allowed to use the result from Exercise 5 below so that you can use familiar rules for computing integrals such as substitution and fundamental theorem of calculus, which we did not prove yet in the setting of the Lebesgue integral. Write $\int_{[0,\infty]}$ appropriately as limit of $\int_{[\varepsilon,R]}$; use convergence theorems correctly to justify.)

4*. Let μ denote Lebesgue measure on $\mathbb{R}$. Suppose $E \subset \mathbb{R}$ is measurable with $\mu(E) > 0$. Prove that $E + E = \{a + b \,:\, a, b \in E\} \subset \mathbb{R}$ contains an open interval. *Hint:* Consider the function $f(x) = \int \mathbf{1}_E(y)\mathbf{1}_E(x - y)d\mu(y)$.

5*. Let $f : [a, b] \to \mathbb{R}$ be a Riemann integrable[1] function. Prove that f is Lebesgue integrable on $[a, b]$ and that the Riemann integral of f is equal to the Lebesgue integral of f.
Hint: First try this yourself and if necessary, refer to Ch. 2, Thm. 1.5 in the book.

[1] Here you may define Riemann integrals via Riemann sums or upper and lower sums, whichever you prefer.