Math 629, Spring 2020 (Roos) – Homework 5.
Due Monday, March 9.

(Problems with an asterisk (*) are optional; problems with two asterisks (**) are optional and may be more challenging.)

1*. Let $p \in [1, \infty]$. Show that there exists a function $f \in L^p(\mathbb{R}^d)$ such that $f \notin L^q(\mathbb{R}^d)$ for all $q \in [1, \infty]$, $p \neq q$.

2. Let (X, Σ, μ) be a σ–finite measure space.
(a) Suppose that $\mu(X) < \infty$. Then $L^p(\mu) \subset L^q(\mu)$ for all $1 \leq q \leq p \leq \infty$.
(b) Suppose that there exists $c > 0$ such that if $\mu(E) < c$, then $\mu(E) = 0$. Then $L^p(\mu) \subset L^q(\mu)$ for all $1 \leq p \leq q \leq \infty$.

3. Suppose that $f \in L^1(\mathbb{R}^d)$ and $\lambda_1, \ldots, \lambda_d > 0$. Define

$$g(x) = f(\lambda_1^{-1} x_1, \ldots, \lambda_d^{-1} x_d).$$

Show that $g \in L^1(\mathbb{R}^d)$ and that

$$\int g = \Big(\prod_{j=1}^{d} \lambda_j \Big) \int f.$$

4. Let $f \in L^p(\mu)$ for some $1 \leq p < \infty$. Prove that for every $\lambda > 0$ we have

$$\mu(\{x : |f(x)| > \lambda\})^{1/p} \leq \lambda^{-1} \|f\|_p.$$

5. Let $1 \leq p < \infty$. Let $(f_n)_n$ be a sequence in $L^p(\mu)$ that converges to some $f \in L^p(\mu)$ (in L^p-norm).

(a) Prove that for every $\varepsilon > 0$ we have

$$\lim_{n \to \infty} \mu(\{x : |f_n(x) - f(x)| > \varepsilon\}) = 0.$$

(b) Construct a sequence $(f_n)_n$ convergent in $L^p(\mathbb{R}^d)$ such that $\lim_{n\to\infty} f_n(x)$ does not exist for a.e. $x \in \mathbb{R}^d$.