**Math 629, Spring 2020 (Roos) – Homework 6.**
Due Monday, March 23.

**Important:** Please submit your homework as a compressed pdf file (< 1 MB, scans of handwritten work are okay, use an appropriate app) online to `jroos@math.wisc.edu` with subject line `Math 629 - Homework 6`.

(Problems with an asterisk (*) are optional; problems with two asterisks (**) are optional and may be more challenging.)

**1.** Define for each $d \geq 1$,

$$c_d = \int_{\mathbb{R}^d} e^{-\pi|x|^2} dx \in (0, \infty).$$

(a) Use polar coordinates to prove that $c_2 = 1$.
(b) Use Fubini's theorem and (a) to prove that $c_d = 1$ for all $d \geq 1$.
(c) Use polar coordinates and (b) to explicitly compute the Lebesgue measure of the unit ball $\{x \in \mathbb{R}^d : |x| \leq 1\}$ in $\mathbb{R}^d$ for all $d \geq 2$. Express the result in terms of powers of $\pi$ and the $\Gamma$-function

$$\Gamma(s) = \int_0^\infty t^{s-1} e^{-t} dt \quad (s > 0)$$

Notes: Here $|x| = (\sum_{i=1}^d |x_i|^2)^{1/2}$. For the purpose of this exercise you may assume that $\pi$ is *defined* as the Lebesgue measure of the set $\{x \in \mathbb{R}^2 : |x| \leq 1\}$.

**2.** Let $f, g \in L^1(\mathbb{R}^d)$. The *convolution* of $f$ and $g$ is the function $f * g$ defined by

$$(f * g)(x) = \int_{\mathbb{R}^d} f(x-y) g(y) dy$$

(a) Show that $(f * g)(x)$ is well-defined for a.e. $x \in \mathbb{R}^d$. That is, show that for a.e. $x \in \mathbb{R}^d$, the function $y \mapsto f(x-y)g(y)$ is integrable (in particular, show that it is measurable).
(b) Verify that the following properties hold for every $f, g, h \in L^1(\mathbb{R}^d)$:

$$f * g = g * f, \quad (f * g) * h = f * (g * h),$$
$$\|f * g\|_1 \leq \|f\|_1 \|g\|_1$$

(c) Show that there does not exist a function $\delta \in L^1(\mathbb{R}^d)$ such that

$$f * \delta = f \text{ for all } f \in L^1(\mathbb{R}^d)$$

**3.** Let $p \in [1, \infty)$ and $(f_n)_{n \in \mathbb{N}} \subset L^p(\mathbb{R}^d)$ such that $\|f_n\|_p \leq n^{-2}$ for all $n \in \mathbb{N}$. Does $(f_n)_{n \in \mathbb{N}}$ necessarily converge pointwise a.e. ? (Proof or counterexample.)

*(Turn the page).*

**4.** (a) Prove the inequality

$$\int_{\mathbb{R}^3} |f(x,y)g(y,z)h(z,x)| d(x,y,z) \leq \|f\|_2 \|g\|_2 \|h\|_2$$

for $f, g, h \in L^2(\mathbb{R}^2)$.

(b) Let $E \subset \mathbb{R}^3$ be a measurable set and suppose that the projections

$$E_1 = \{(y,z) : (x,y,z) \in E\},$$
$$E_2 = \{(x,z) : (x,y,z) \in E\},$$
$$E_3 = \{(x,y) : (x,y,z) \in E\}$$

are measurable subsets of $\mathbb{R}^2$. Use (a) to derive an upper bound on the measure of $E$ in terms of the measures of $E_1, E_2, E_3$.

**5**.** Let $X$ be a metric space and $\mu$ a Borel measure on $X$. Let $C_c(X)$ denote the set of continuous functions on $X$ that are supported in a compact set (i.e. $\overline{\{x : f(x) \neq 0\}} \subset X$ is compact). The point of this exercise is to prove that $C_c(X)$ is dense in $L^p(\mu)$ for every $p \in [1, \infty)$.

(a) Let $A \subset X$ be a closed set and $f : A \to \mathbb{C}$ a continuous function. Show that there exists a continuous function $f^\sharp : X \to \mathbb{C}$ such that $f^\sharp|_A = f$ and $\|f^\sharp\|_\infty = \|f\|_\infty$.

*Hint:* First use the metric to construct continuous functions that separate disjoint closed sets: given $A, B \subset X$ disjoint and closed there exists a continuous $g : X \to [0,1]$ such that $g = 0$ on $A$ and $g = 1$ on $B$.

(b) Let $p \in [1, \infty)$ and $f \in L^p(\mu)$. Show that for every $\varepsilon > 0$ there exists $g \in C_c(X)$ such that $\|f - g\|_{L^p(\mu)} \leq \varepsilon$.

*Hint:* First let $p = 1$. Use (a) and Lusin's theorem (see Homework 3, Exercise 5; the statement continues to hold in the metric space setting).

(c) Show that $C_c(\mathbb{R})$ is not dense in $L^\infty(\mathbb{R})$.

**6*.** A measure space is called *complete* if every subset of a nullset (a measurable set of measure zero) is measurable. Let $(X, \Sigma, \mu)$ be a measure space. We define its *completion* $(X, \overline{\Sigma}, \overline{\mu})$ as follows: $\overline{\Sigma}$ is the collection of all sets of the form $E \cup N$ where $E \in \Sigma$ and $N \subset F$ for some $F \in \Sigma$ with $\mu(F) = 0$. Define $\overline{\mu}(E \cup N) = \mu(E)$.

(i) Prove that $\overline{\Sigma}$ is a $\sigma$–algebra (called completion of $\Sigma$).

(ii) Prove that $\overline{\mu}$ is a measure.

(iii) Prove that the Lebesgue $\sigma$-algebra on $\mathbb{R}^d$ is the completion of the Borel $\sigma$-algebra.

(iv) Let $\mu_d$ denote the Lebesgue measure on $\mathbb{R}^d$ and $\Sigma_d$ the Lebesgue $\sigma$-algebra on $\mathbb{R}^d$. Prove that the completion of the product measure space $(\mathbb{R}^{d_1}, \Sigma_{d_1}, \mu_{d_1}) \times (\mathbb{R}^{d_2}, \Sigma_{d_2}, \mu_{d_2})$ as defined in class coincides with $(\mathbb{R}^{d_1+d_2}, \Sigma_{d_1+d_2}, \mu_{d_1+d_2})$.