Math 629, Spring 2020 (Roos) – Homework 7.
Due Monday, April 6.

Important: Please submit your homework as a compressed pdf file (< 1 MB, scans of handwritten work are okay, use an appropriate app) online to `jroos@math.wisc.edu` with subject line `Math 629 - Homework 7`.

(Problems with an asterisk (*) are optional; problems with two asterisks (**) are optional and may be more challenging.)

1. Let $\mathcal{R}$ be a collection of measurable sets in $\mathbb{R}^d$ with *bounded eccentricity.* That is, there exists $A \in (0, \infty)$ such that for every $R \in \mathcal{R}$ exists an open ball B such that $R \subset B$ and $\mu(B) \leq A\mu(R)$ (μ denotes Lebesgue measure). For measurable $f : \mathbb{R}^d \to \mathbb{C}$ define

$$M_{\mathcal{R}} f(x) = \sup_{x \in R \in \mathcal{R}} \frac{1}{\mu(R)} \int_R |f|. \tag{1}$$

Prove that there exists $C \in (0, \infty)$ such that for all $f \in L^1(\mathbb{R}^d)$,

$$\sup_{\lambda > 0} \lambda \mu(\{M_{\mathcal{R}} f > \lambda\}) \leq C \|f\|_{L^1(\mathbb{R}^d)}. \tag{2}$$

2. For $f \in L^1_{\mathrm{loc}}(\mathbb{R})$, define the (centered) Hardy-Littlewood maximal function

$$Mf(x) = \sup_{r>0} \frac{1}{2r} \int_{-r}^{r} |f(x-t)| dt, \quad (x \in \mathbb{R})$$

(a) Fix real numbers $a < b$. Compute $M\mathbf{1}_{[a,b]}$ explicitly.
(b) Suppose that $Mf \in L^1(\mathbb{R})$ for some $f \in L^1_{\mathrm{loc}}(\mathbb{R})$. Prove that $f = 0$ a.e.

3. Let $E \subset [0, 1]$ be measurable and $\alpha \in (0, 1)$. Assume that for every interval $I \subset [0, 1]$ we have $\mu(E \cap I) \geq \alpha \cdot \mu(I)$. Prove that $\mu(E) = 1$. (μ denotes Lebesgue measure.)

4. Let $f \in L^1_{\mathrm{loc}}(\mathbb{R}^d)$. $x \in \mathbb{R}^d$ is called *Lebesgue point* of f if

$$\lim_{\mu(B) \to 0, x \in B} \frac{1}{\mu(B)} \int_B |f(y) - f(x)| d\mu(y) = 0,$$

where B denotes a Euclidean ball.
(i) Prove that if f is continuous at x, then x is a Lebesgue point of f.
(ii) Prove that almost every $x \in \mathbb{R}^d$ is a Lebesgue point of f.

(Turn the page.)

5*. Let $\mathcal{R}$ be the collection of axis-parallel rectangles in $\mathbb{R}^2$. With $M_{\mathcal{R}}$ defined as in (1), show that (2) does not hold. That is, show that for every $n \in \mathbb{N}$ there exists $f \in L^1(\mathbb{R}^2)$ so that

$$\sup_{\lambda>0} \lambda\mu(\{M_{\mathcal{R}}f > \lambda\}) \geq n\|f\|_{L^1(\mathbb{R}^2)}.$$

6*. Let φ be a non-negative integrable function on $\mathbb{R}$ which is even (that is, $\varphi(x) = \varphi(-x)$ for all $x \in \mathbb{R}$) and non-increasing on $[0, \infty)$. For $t > 0$ we let $\varphi_t(x) = t^{-1}\varphi(t^{-1}x)$. Prove that for a.e. $x \in \mathbb{R}$ and every integrable function f we have

$$\sup_{t>0} |f * \varphi_t(x)| \leq \|\varphi\|_1 Mf(x),$$

where Mf denotes the centered Hardy-Littlewood maximal function as defined in Problem 2. *Hint:* First prove it if φ is a simple function.