Math 629, Spring 2020 (Roos) – Homework 8.
Due Monday, April 13.

Important: Please submit your homework as a compressed pdf file (< 1 MB, scans of handwritten work are okay, use an appropriate app) online to `jroos@math.wisc.edu` with subject line `Math 629 - Homework 8`.

(Problems with an asterisk (*) are optional; problems with two asterisks (**) are optional and may be more challenging.)

1. Let $f \in L^1(\mathbb{R}^d)$. Define the *Fourier transform* of f by

$$\widehat{f}(\xi) = \int_{\mathbb{R}^d} f(x) e^{-2\pi i x\cdot\xi} dx \quad (\xi \in \mathbb{R}^d)$$

(Here $x \cdot \xi = \sum_{i=1}^d x_i \xi_i$.) Observe that $\widehat{f}(\xi)$ is well-defined, since $|e^{-2\pi i x\cdot\xi}| = 1$ for every $x, \xi \in \mathbb{R}^d$, so $x \mapsto f(x)e^{-2\pi i x\cdot\xi}$ is integrable for every $\xi \in \mathbb{R}^d$.
(i) Show that $\widehat{f} : \mathbb{R}^d \to \mathbb{C}$ is a continuous function and $\|\widehat{f}\|_\infty \leq \|f\|_1$.
(ii) Show that for every $f, g \in L^1(\mathbb{R}^d)$ we have

$$\widehat{f * g} = \widehat{f} \cdot \widehat{g}.$$

(iii) Define *translation* of f by $T_y f(x) = f(x-y)$ for $y \in \mathbb{R}^d$, *modulation* of f by $M_\xi f(x) = e^{2\pi i x\cdot\xi} f(x)$ for $\xi \in \mathbb{R}^d$ and *dilation* of f by $D_\delta^p f(x) = \delta^{\frac{d}{p}} f(\delta x)$ for $\delta > 0$ and $p \in [1, \infty]$. Prove the following properties:

$$\widehat{T_y f} = M_{-y}\widehat{f}, \quad \widehat{M_\xi f} = T_\xi \widehat{f}, \quad \widehat{D_\delta^p f} = D_{\delta^{-1}}^{p'} \widehat{f}$$

(Here $\frac{1}{p} + \frac{1}{p'} = 1$.)
(iv) Let $g(x) = e^{-\pi|x|^2}$ (we know that $g \in L^1(\mathbb{R}^d)$). Prove that $\widehat{g} = g$.
Hint: First prove it for $d = 1$ and then use Fubini.
(v*) Prove that if f is compactly supported, then $\widehat{f}$ cannot be compactly supported.

2. Let $K \in L^1(\mathbb{R}^d)$ be bounded and supported on a bounded set with $\int K = 1$. Define $K_\delta(x) = \delta^{-d} K(\delta^{-1}x)$ for every $\delta > 0$. Show that $(K_\delta)_{\delta>0}$ is an approximation of identity in the sense defined in the lecture.

3. Let $E \subset \mathbb{R}$ be a measurable set with positive Lebesgue measure.
(i) Show that there exist $x \in \mathbb{R}, y > 0$ such that $x, x+y, x+2y \in E$.
(ii*) Let k be a positive integer. Show that there exist $x \in \mathbb{R}, y > 0$ such that $x, x+y, x+2y, \ldots, x+ky \in E$.

(Turn the page.)

4. Let $E \subset \mathbb{R}$ be measurable with $m(E) > 0$. Does there exist a sequence $(s_n)_{n\in\mathbb{N}}$ such that the complement of $\bigcup_{n\in\mathbb{N}}(s_n + E)$ has measure zero? (Prove or disprove.)

5*. In this exercise we introduce a variant of the Hardy-Littlewood maximal function and use it to give an alternative proof of the weak (1,1) bound for the Hardy-Littlewood maximal function. Fix a real number s. By $\mathcal{D}_s$ we denote the collection of intervals of the form $[2^k(\ell + s), 2^k(\ell + s + 1))$ with $k, \ell \in \mathbb{Z}$. These are called *dyadic intervals* (with shift s). Observe that every two dyadic intervals $I, J \subset \mathcal{D}_s$ have the property that they are either disjoint or contained in each other. Given $f \in L^1(\mathbb{R}^d)$, define the *dyadic maximal function*

$$M_s f(x) = \sup_{I\in\mathcal{D}_s, x\in I} \frac{1}{|I|}\int_I |f| \quad (x \in \mathbb{R})$$

(i) Show that for every $\lambda > 0$, the set $\{M_0 f > \lambda\} \subset \mathbb{R}$ can be written as a union over pairwise disjoint dyadic intervals in $\mathcal{D}_0$.

(ii) Prove that there exists a constant $C > 0$ such that for every $\lambda > 0$ and every $f \in L^1(\mathbb{R})$ we have[1]

$$\mu(\{M_0 f > \lambda\}) \le C\lambda^{-1}\|f\|_1.$$

(iii) Observe that (ii) also holds with M_s in place of M_0 for every $s \in \mathbb{R}$.

(iv) Show that there exists a constant $c > 0$ such that for every interval $I \subset \mathbb{R}$ there exists an interval $J \in \mathcal{D}_0 \cup \mathcal{D}_{1/3} \cup \mathcal{D}_{2/3}$ such that $I \subset J$ and $|J| \le c|I|$.

(v) Let M denote the Hardy-Littlewood maximal function as defined in the lecture. Show that there exists $c > 0$ such that

$$Mf \le c(M_0 f + M_{1/3} f + M_{2/3} f)$$

In particular, $\mu(\{Mf > \lambda\}) \le c'\lambda^{-1}\|f\|_1$ for every $\lambda > 0$.

[1] You are not allowed to use the weak (1,1) bound for the Hardy-Littlewood maximal function seen in the lecture!