Math 629, Spring 2020 (Roos) – Homework 9.
Due Monday, April 20.

Important: Please submit your homework as a compressed pdf file (< 1 MB, scans of handwritten work are okay, use an appropriate app) online to `jroos@math.wisc.edu` with subject line `Math 629 - Homework 9`.

(Problems with an asterisk (*) are optional; problems with two asterisks (**) are optional and may be more challenging.)

1. Let $(X, \|\cdot\|)$ denote a normed vector space (over $\mathbb{R}$ or $\mathbb{C}$).
(i) Show that there exists an inner product $\langle \cdot, \cdot \rangle$ on X such that $\langle x, x\rangle = \|x\|^2$ if and only if for all $x, y \in X$,

$$2\|x\|^2 + 2\|y\|^2 = \|x+y\|^2 + \|x-y\|^2.$$

(ii) Show that L^p (where $p \in [1, \infty]$) can be equipped with an inner product so that $\langle f, f\rangle = \|f\|_p^2$ if and only if $p = 2$.

2. By $C_c^\infty(\mathbb{R}^d)$ let us denote the space of smooth functions $\varphi : \mathbb{R}^d \to \mathbb{C}$ that have a compact support $\mathrm{supp}(f) = \overline{\{x \in \mathbb{R}^d \,:\, f(x) \neq 0\}}$.
(i) $C_c^\infty(\mathbb{R}^d)$ contains non-trivial functions: consider $\varphi(x) = \mathbf{1}_{|x|<1} e^{-(1-|x|^2)^{-1}}$. Show that $\varphi \in C_c^\infty(\mathbb{R}^d)$.
(ii) Prove that $C_c^\infty(\mathbb{R}^d)$ is a dense subspace of $L^p(\mathbb{R}^d)$ for all $p \in [1, \infty)$.
(iii*) Looking at the function $f = 1 \in L^\infty(\mathbb{R}^d)$ shows that (ii) fails for $p = \infty$. Determine the closure of $C_c^\infty(\mathbb{R}^d)$ in $L^\infty(\mathbb{R}^d)$.

3. Suppose that $f \in L^1([0,1])$. For $n \in \mathbb{Z}$ let $a_n = \int_0^1 f(x) e^{-2\pi i x n} dx \in \mathbb{C}$. For each $x \in [0,1]$ and $r \in [0,1)$ define

$$\mathcal{A}(x, r) = \sum_{n \in \mathbb{Z}} a_n r^{|n|} e^{2\pi i n x}.$$

(Observe the series is absolutely summable if $r \in [0,1)$.)
Show that for almost every $x \in [0,1]$, $\mathcal{A}(x, r)$ converges to $f(x)$ as $r \to 1$.

4. Let $f \in L^p(\mathbb{R}^d)$ for $p \in [1, \infty)$. Show that

$$\|f\|_p = \Big(\int_0^\infty p \lambda^{p-1} \mu(\{|f| > \lambda\}) d\lambda \Big)^{1/p}.$$

(Turn the page.)

5.** Let (X, Σ, μ) be a σ-finite measure space. For $p \in [1, \infty)$ denote by $L^{p,\infty}$ (weak-L^p) the set of measurable functions $f : X \to \mathbb{C}$ so that

$$[f]_{p,\infty} := \sup_{\lambda > 0} \lambda \mu(\{|f| > \lambda\})^{1/p} < \infty.$$

We also write $L^{\infty,\infty} = L^\infty$. As in the case $p = 1$ seen in the lecture observe that this defines a quasi-norm and $[f]_{p,\infty} \leq \|f\|_p$. Let $p, q \in [1, \infty]$ and let T_1 denote a bounded linear operator $L^p \to L^{p,\infty}$ and T_2 a bounded linear operator $L^q \to L^{q,\infty}$ so that $T_1|_{L^p \cap L^q} = T_2|_{L^p \cap L^q}$. Let $\theta \in (0, 1)$ and p_θ be defined by $\frac{1}{p_\theta} = \frac{\theta}{p} + \frac{1-\theta}{q}$. Prove that there exists a bounded linear extension T of T_1, T_2 mapping $L^{p_\theta} \to L^{p_\theta}$.

Hints: It helps to work with dense subspaces as seen in the lecture. Then T, T_1, T_2 are mostly denoted by the same letter, say T. Use Problem 4 to start estimating $\|Tf\|_{p_\theta}$. Cut functions into pieces with large and small values and use Chebyshev's inequality. Treat the (easier) case when p or q is ∞ separately.

6*. These are sample applications of the result in Problem 5:

(i) Let M denote the Hardy-Littlewood maximal operator on $\mathbb{R}^d$ seen in the lecture. Prove that M is bounded $L^p \to L^p$ for all $p \in (1, \infty]$.

(ii) Let $\mathcal{F}$ denote the Fourier transform, initially defined on $L^1(\mathbb{R}^d)$. In the lecture we saw how to extend $\mathcal{F}$ from a dense subspace to $L^2(\mathbb{R}^d)$ via Plancherel's theorem. Prove that for each $p \in (1, 2)$, $\mathcal{F}$ extends to a bounded linear operator $L^p(\mathbb{R}^d) \to L^{p'}(\mathbb{R}^d)$. That is, for each $p \in (1, 2)$ show that there exists $C \in (0, \infty)$ so that $\|\mathcal{F}f\|_{p'} \leq C\|f\|_p$ for a suitable dense class of f. Here $\frac{1}{p} + \frac{1}{p'} = 1$.

(iii**) Prove that (ii) fails for $p > 2$: the Fourier transform cannot be extended to $L^p(\mathbb{R}^d)$ for $p > 2$.