Math 192r, Problem Set #14: Solutions

1. *Use the recurrence for $p(n)$ to compute the last digit of $p(n)$ for every n between 1 and 1000. Can you make any conjectures about the relationship between the last digit of n and the last digit of $p(n)$?*

   Here's a Maple program that does this:

```
F := proc(n) option remember; local total, k;
if n=0 then 1; elif n<0 then 0; else total := 0;
k := 1; while k*(3*k+1)/2 <= n do
total := total - (-1)^k*F(n-k*(3*k+1)/2): k := k+1: od:
k := -1; while k*(3*k+1)/2 <= n do
total := total - (-1)^k*F(n-k*(3*k+1)/2): k := k-1: od:
total mod 10; fi: end;
```

   We then create a matrix to keep track of how often it happens that n ends with the digit i while $p(n)$ ends with the digit j (for i, j between 0 and 9), and print out its entries:

```
for i from 0 to 9 do for j from 0 to 9 do a[i,j]:=0: od: od:
for n from 1 to 1000 do k: = F(n);
       a[n mod 10, k mod 10] := a[n mod 10, k mod 10] + 1; od:
for i from 0 to 9 do seq(a[i,j],j=0..9) od;
```

   This results in the output

```
14, 7, 13, 12, 3, 5, 12, 15, 9, 10
9, 11, 14, 9, 8, 9, 10, 13, 7, 10
3, 14, 12, 14, 10, 8, 8, 12, 6, 13
8, 9, 12, 9, 8, 17, 5, 13, 9, 10
49, 0, 0, 0, 0, 51, 0, 0, 0, 0
5, 12, 10, 4, 15, 7, 13, 13, 8, 13
8, 14, 11, 15, 7, 9, 8, 6, 5, 17
9, 10, 6, 9, 16, 10, 9, 9, 12, 10
10, 9, 16, 8, 11, 11, 11, 13, 5, 6
48, 0, 0, 0, 0, 52, 0, 0, 0, 0
```

from which we conjecture that when n ends in 4 or 9, $p(n)$ ends in 0 or 5. That is, if n is 1 less than a multiple of 5, $p(n)$ is a multiple of 5.

(This fact was first noticed and proved by Ramanujan.)

2. *Let $F(0) = 1$ and recursively define $F(n) = F(n-1) + F(n-3) - F(n-6) - F(n-10) + F(n-15) + F(n-21) - - + + \ldots$ for all $n > 0$, where terms of the form $F(n-k)$ are to be ignored once $k > n$. There exists a set S of positive integers such that $F(n)$ equals the number of partitions of n into parts belonging to S. Find S (conjecturally).*

   This property of S is equivalent to $1 - q - q^3 + q^6 + q^{10} - - + + \ldots = \prod_{k \in S}(1 - q^k)$. We note first that we must have $1 \in S$, since if not, every factor in the product would have vanishing coefficient of q^1. Next we have $\prod_{k \in S \setminus \{1\}}(1 - q^k) = (1 - q - q^3 + \ldots)/(1 - q) = 1 + 0q + 0q^2 - q^3 \ldots$. We now note that we must have $2 \notin S$, since otherwise we would have $\prod_{k \in S \setminus \{1\}}(1 - q^k)$ of the form $1 - q^2 \ldots$. On the other hand, we must have $3 \in S$. We now continue with $\prod_{k \in S \setminus \{1,3\}}(1 - q^k) = (1 - q - q^3 + q^6 \ldots)/(1 - q)(1 - q^3) = 1 + 0q + 0q^2 + 0q^3 - q^4 - q^5 - q^7 \ldots$ to conclude that 4 and 5 (but not 6) belong to S. And so on. Empirically, we find that S is just the set of numbers that are not congruent to 2 mod 4.