Math 192r, Problem Set #18
(due 12/4/01)

1. (from unpublished work of Douglas Zare) Let $G_{m,n}$ be the directed graph with vertex set $\{(i,j) \in \mathbf{Z} \times \mathbf{Z} : 0 \leq i \leq m,\ 0 \leq j \leq n\}$, with an arc from (i,j) to (i',j') iff $(j'-j, i'-i)$ is $(1,0)$, $(0,1)$, or $(1,1)$.

   (a) For any legal path P in $G_{m,n}$ from $(0,0)$ to (m,n), define $d(P)$ as the number of diagonal steps in P plus the number of upward steps in P that are followed *immediately* by a rightward step. Show that the number of paths P with $d(P) = k$ is exactly $2^k \binom{m}{k} \binom{n}{k}$.

   (b) Let M be the $(n+1)$-by-$(n+1)$ matrix with rows and columns indexed from 0 through n whose i,jth entry is the total number of paths in $G_{i,j}$ from $(0,0)$ to (i,j). Use the result of part (a) to find the LDU decomposition of M. That is: find square matrices L, D, U such that $LDU = M$, where L (resp. U) is a lower (resp. upper) triangular matrix with 1's on the diagonal and where D is a diagonal matrix (whose diagonal entries are permitted to be different). Use this in turn to compute $\det(M)$.

   (c) Interpret M as the Lindstrom matrix of some directed graph and use this in turn to interpret $\det(M)$ as the number of perfect matchings of some graph H_n. Be explicit about what H_n looks like.

2. Fix positive integers a, b, m, n with $n > m$ and $a + n \leq b$, and let $M(a+1), M(a+2), \ldots, M(b+n-1)$ be arbitrary m-by-m matrices. Show that the n-by-n matrix whose i,jth entry (for $1 \leq i,j \leq n$) is the upper left entry of the product matrix $M(a+i)M(a+i+1)\cdots M(b+j-1)$ has determinant zero.