[Get 75% of students to turn on cameras]

Section 3.5: Mathematical Systems and Proofs

- Write direct and indirect proofs in 3-column format.

(not required this semester)

Questions on 3.5?

Questions about proofs.pdf?

Section 3.6: Propositions over a Universe

- Know basic terminology: Proposition over a universe and the corresponding concepts of tautology, contradiction, equivalence, and implication.
- Determine the truth set of a proposition over a universe.
- Use truth sets to determine if one proposition implies another proposition, or if two propositions are equivalent.

Questions on 3.6?

Propositions over a universe: Suppose p is a function from some universe U to the set {True, False}. Then we call p a proposition over the universe U and we say its truth set T_p is the set $\{x \in U : p(x) = \text{True}\}$.

It's important to distinguish between p (a proposition over a universe) and T_p (the truth set of that proposition), even though they're intimately related, and even though (as we'll see in Chapter 6) some mathematical conventions seem to mistake one for the other!

Suppose p and q are propositions over a universe U.

$p \Rightarrow q$ is equivalent to the assertion that T_p is a subset of T_q.

Also, $p \Leftrightarrow q$ is equivalent to the assertion that T_p equals T_q.

Suppose $p(x)$ is the proposition "x is positive" and $q(x)$ is the proposition "x has a square root" (where U is the set of real numbers). Then $T_p = (0, \infty)$ while $T_q = [0, \infty)$. Since T_p is a subset of T_q but not vice versa, $p \Rightarrow q$ but not vice versa.

The proposition $\neg p$ says "x is not positive"; its truth set is $(-\infty,0]$. Note that $T_{\neg p}$ is the complement of T_p.

Another example: Let $U = \{n \in \mathbb{Z}: n \geq 2\}$, and $p(n)$ be the proposition "n is prime", so that $p(n)$ is the proposition "n is composite". Then $T_p = \{2,3,5,7,\ldots\}$ and $T_{\neg p} = \{4,6,8,9,\ldots\}$.

Note that in both examples $T_{\neg p} = (T_p)^c$.

That is not a coincidence! This equality holds for all propositions p over an arbitrary universe U.

Claim: If p is a proposition over a universe U, $T_{\neg p} = (T_p)^c$.

Proof:

$x \in T_{\neg p} \Leftrightarrow (\neg p)(x)$ is true

$\Leftrightarrow p(x)$ is false

$\Leftrightarrow x \in T_p$ is false

$\Leftrightarrow x \in (T_p)^c$

Since we have shown that for all x in U, x belongs to $T_{\neg p}$ if and only if x belongs to $(T_p)^c$, we have proved $T_{\neg p} = (T_p)^c$.

(Note that if any of those two-headed arrows had been one-headed $\Rightarrow$–arrows, then all we'd be able to conclude is that $x \in T_{\neg p} \Rightarrow x \in (T_p)^c$, and all we'd learn from that is that $T_{\neg p}$ is a SUBSET of $(T_p)^c$. To get equality of the sets, we need bidirectional equivalence of the associated propositions.)

Group work: 3.6.2 (8 minutes), using Google Docs (include six links in chat).

Over the universe of positive integers, define

$$\begin{aligned} p(n)&: \quad n \text{ is prime and } n < 32. \\ q(n)&: \quad n \text{ is a power of 3.} \\ r(n)&: \quad n \text{ is a divisor of 27.} \end{aligned}$$

(a) What are the truth sets of these propositions?

(b) Which of the three propositions implies one of the others?

(Recall: A prime is a whole number bigger than 1 whose only divisors are 1 and itself; 1 is not considered prime. If $k > 0$, the powers of k are $k^0 = 1$, $k^1 = k$, k^2, k^3, …)

..?..

$T_p = \{2,3,5,7,11,13,17,19,23,29,31\}$

$T_q = \{1,3,9,27,81,\ldots\}$

$T_r = \{1,3,9,27\}$

T_r is a subset of T_q, so $r \Rightarrow q$.

Don’t make the mistake of writing things like

“$p(n) = \{2,3,5,7,11,13,17,19,23,29,31\}$”

The left hand side is a *proposition*; the right hand side is a *set*. They aren’t equal.

Similarly, don’t write $p \cap q$ (you can’t take the intersection of two propositions) of $T_p \wedge T_q$ (you can’t take the conjunction of two sets.

We can combine propositions over U to create new propositions over U.

For instance, $p \wedge q$ is the proposition whose truth value for each n is given by "$p(n) \wedge q(n)$", which is true for some n and false for others.

(We could write "$(p \wedge q)(n) = p(n) \wedge q(n)$", if that's helpful.)

In particular, for the propositions p and q discussed above,

"$p(1) \wedge q(1)$" is of the form "$0 \wedge 1$", which is False;

"$p(2) \wedge q(2)$" is of the form "$1 \wedge 0$", which is False;

"$p(3) \wedge q(3)$" is of the form "$1 \wedge 1$", which is True; etc.

So $(p \wedge q)(1)$ = False, $(p \wedge q)(2)$ = False, $(p \wedge q)(3)$ = True, etc., and $T_{p \wedge q} = \{3\}$.

If we like we could make a table, something like a truth table:

n	$p(n)$	$q(n)$	$(p \wedge q)(n)$
1	0	1	0
2	1	0	0
3	1	1	1
4	0	0	0
5	1	0	0
…	…	…	…

Note that in this example $T_{p \wedge q} = T_p \cap T_q$. That is not a coincidence! This equality holds for all propositions p and q over an arbitrary universe U.

Let's see how we might prove this (and in so doing lay the groundwork for your solution to a problem from the next homework).

Claim: If p and q are propositions over a universe U, $T_{p \wedge q} = T_p \cap T_q$.

Proof:

$$
\begin{aligned}
x \in T_{p \wedge q} &\Leftrightarrow (p \wedge q)(x) \text{ is true} \\
&\Leftrightarrow p(x) \text{ is true and } q(x) \text{ is true} \\
&\Leftrightarrow x \in T_p \text{ and } x \in T_q \\
&\Leftrightarrow x \in T_p \cap T_q
\end{aligned}
$$

Since x belongs to $T_{p \wedge q}$ if and only if x belongs to $T_p \cap T_q$, $T_{p \wedge q} = T_p \cap T_q$.

Likewise:

Claim: If p and q are propositions over some universe U, $T_{p \vee q} = T_p \cup T_q$.

Group work: Prove it (8 minutes) with Zoom whiteboard.

If formatting is hard, write things like "T-sub-(p or q) = T_p union T_q".

..?..

Proof:

$x \in T_{p \vee q} \Leftrightarrow (p \vee q)(x)$ is true

$\Leftrightarrow p(x)$ is true or $q(x)$ is true

$\Leftrightarrow x \in T_p$ or $x \in T_q$

$\Leftrightarrow x \in T_p \cup T_q$

Since x belongs to $T_{p \vee q}$ if and only if x belongs to $T_p \cup T_q$, $T_{p \vee q} = T_p \cup T_q$.

Compare Google Docs with Zoom whiteboard (chat poll)