[Get 75% of students to turn on cameras]

Are you able to see the points breakdown and other comments on Blackboard?

Questions on sections 3.1 through 3.6?

Thoughts on break-out rooms?

Section 3.7: Mathematical Induction

- Prove statements of the form "For all positive integers n, $p(n)$ is true" using induction.

Principle of mathematical induction: If the propositions

P_1,

$P_1 \Rightarrow P_2$,

$P_2 \Rightarrow P_3$,

$P_3 \Rightarrow P_4$,

etc.

are all true, then P_n is true for all n.

That is, if P_1 is true, and if for all $n \geq 1$ $P_n \Rightarrow P_{n+1}$ is true,

then P_n is true for all $n \geq 1$.

~~Group work~~: 3.7.2 (12 minutes)

(http://jamespropp.org/2190/3.7.2.png)

Prove that if $n \geq 1$, then $1(1!) + 2(2!) + \cdots + n(n!) = (n+1)! - 1$.

..?..

Theorem: If $n \geq 1$, $1(1!) + 2(2!) + \ldots + n(n!) = (n+1)! - 1$.

Proof: Use mathematical induction.

Base case: $1(1!) = 1 = 2 - 1 = (1+1)! - 1$.

Induction step: Suppose n is a number for which the formula holds, so that $1(1!) + 2(2!) + \ldots + n(n!) = (n+1)! - 1$. Then

$$
\begin{aligned}
&1(1!) + 2(2!) + \ldots + n(n!) + (n+1)(n+1)! \\
&\quad = ((n+1)! - 1) + (n+1)(n+1)! \\
&\quad = (1)(n+1)! + (n+1)(n+1)! - 1 \\
&\quad = (1+n+1)\,(n+1)! - 1 \\
&\quad = (n+2)\,(n+1)! - 1 \\
&\quad = (n+2)! - 1
\end{aligned}
$$

so that $n+1$ is a positive integer for which the formula is true.

Since we have verified the base case ($n=1$) and the induction step, the formula holds for all positive integers.

Reindexed version of mathematical induction (sometimes more convenient): If P_1 is true, and if for all $n \geq 2$ $P_{n-1} \Rightarrow P_n$ is true, then P_n is true for all $n \geq 1$.

~~Group work~~: 3.7.5 (12 minutes) (http://jamespropp.org/2190/3.7.5.png)

Use mathematical induction to show that for $n \geq 1$,

$$\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \cdots + \frac{1}{n(n+1)} = \frac{n}{n+1}.$$

Theorem: If $n \geq 1$, $1/(1)(2) + 1/(2)(3) + \ldots + 1/(n)(n+1) = n/(n+1)$.

Proof: Use mathematical induction (re-indexed for convenience, as you'll see).

Base case: $1/(1)(2) = 1/2$.

Induction step: Suppose $n-1$ is a positive integer for which the formula is true, so that

$$1/(1)(2) + 1/(2)(3) + \ldots + 1/(n-1)(n) = (n-1)/n.$$

Then

$$\begin{aligned} &1/(1)(2) + \ldots + 1/(n-1)(n) + 1/(n)(n+1) \\ &\quad = (n-1)/n + 1/(n)(n+1) \\ &\quad = (n-1)(n+1)/(n)(n+1) + 1/(n)(n+1) \\ &\quad = ((n-1)(n+1)+1)/(n)(n+1) \\ &\quad = n^2 / (n)(n+1) \\ &\quad = n/(n+1) \end{aligned}$$

so that n is a positive integer for which the formula is true.

Since we have verified the base case and the induction step, the formula holds for all positive integers.

Principle of strong induction: If the propositions

P_1,

$P_1 \Rightarrow P_2$,

$P_1 \wedge P_2 \Rightarrow P_3$,

$P_1 \wedge P_2 \wedge P_3 \Rightarrow P_4$,

etc.

are all true, then P_n is true for all n. That is, if P_1 is true, and if for all $n \geq 1$ $(P_1 \wedge P_2 \wedge \ldots \wedge P_n) \Rightarrow P_{n+1}$ is true, then P_n is true for all $n \geq 1$.

Re-indexed version of strong induction: If P_1 is true, and if for all $n \geq 2$ $(P_1 \wedge P_2 \wedge \ldots \wedge P_{n-1}) \Rightarrow P_n$ is true, then P_n is true for all $n \geq 1$.

Application: Every positive integer can be written as a sum of distinct powers of 2.

Proof by strong induction: 1 is a power of 2 (namely 2^0). Now take $n \geq 1$, and suppose we know that every number from 1 to n–1 is a sum of distinct powers of 2; we wish to prove that the same is true for n.

Case 1: If n is even, let $m = n/2$; since $m < n$, m can be written as a sum of distinct powers of 2. Just double them all and we get n expressed as a sum of distinct powers of 2. Moreover, all those powers are bigger than 1.

Case 2: If n is odd, let $m = n$–1 (an even number). As we saw in case 1, m can be written as a sum of distinct powers of 2 all bigger than 1; including 1 in the sum gives us n expressed as a sum of distinct powers of 2.

Course-of-values induction is the same idea as strong induction, but slightly more general.

E.g., course-of-values induction tells us that if P_4 is true, and P_5 is true, and $(P_4 \wedge P_5 \wedge \ldots \wedge P_n) \Rightarrow P_{n+1}$ is true for all $n \geq 5$, then P_n is true for all $n \geq 4$.

Other questions on 3.7?