[Get 75% of students to turn on cameras]

Section 3.8: Quantifiers

- Know basic terminology: existential quantifier, universal quantifier
- Write statements symbolically using quantifiers. Translate symbolic statements involving quantifiers into English.
- Write negations of quantified statements using the rules given in Section 3.8.3.

Suppose p is a proposition over the universe U.

What does $(\forall x)_U\, p(x)$ mean?

..?..

For all x in U, $p(x)$ is true.

The upside-down "A" stands for **A**ll.

What does $(\exists x)_U\, p(x)$ mean?

..?..

There exists an x in U for which $p(x)$ is true.

The backwards "E" stands for **E**xists.

Another way of saying it:

$(\forall x)_U \, p(x)$ means $T_p = U$.

$(\exists x)_U \, p(x)$ means $T_p \neq \emptyset$.

If the universe U is finite, say $U = \{a_1, a_2, \ldots, a_n\}$, then

$$(\forall x)_U \, p(x)$$

is a short way of saying

$$p(a_1) \wedge p(a_2) \wedge \ldots \wedge p(a_n),$$

while

$$(\exists x)_U \, p(x)$$

is a short way of saying

$$p(a_1) \vee p(a_2) \vee \ldots \vee p(a_n).$$

We pronounce $(\forall x)_U\, p(x)$ as "For all x belonging to U, $p(x)$ is true", and we pronounce $(\exists x)_U\, p(x)$ as "There exists an x belonging to U such that $p(x)$ is true."

(Don't leave out the "such that"!)

One of De Morgan's Laws says that

$$\neg (p \wedge q) \Leftrightarrow \neg p \vee \neg q$$

which implies

$$\neg (p \wedge q \wedge r) \Leftrightarrow \neg p \vee \neg q \vee \neg r$$

etc. The version of this that applies to propositions over a universe $\{a_1, a_2, \ldots, a_n\}$ is

$$\neg (p(a_1) \wedge \ldots \wedge p(a_n)) \Leftrightarrow \neg p(a_1) \vee \ldots \vee \neg p(a_n)$$

so that

$$\neg (\forall x)_U\, p(x) \Leftrightarrow (\exists x)_U\, (\neg p(x));$$

similarly

$$\neg (\exists x)_U\, p(x) \Leftrightarrow (\forall x)_U\, (\neg p(x)).$$

Questions on 3.8?

Group work: 3.8.1

(http://jamespropp.org/2190/3.8.1.png)

Let $C(x)$ be "x is cold-blooded," let $F(x)$ be "x is a fish," and let $S(x)$ be "x lives in the sea."

(a) Translate into a formula: Every fish is cold-blooded.

(b) Translate into English: $(\exists x)(S(x) \wedge \neg F(x))$.

(c) Translate into English: $(\forall x)(F(x) \rightarrow S(x))$.

..?..

Let's do (c) first.

(c):

..?..

"All fish live in the sea."

(a):

..?..

$(\forall x)\ (F(x) \rightarrow C(x))$

(b):

..?..

"There's a sea-dwelling creature that isn't a fish."

Double quantifiers: Suppose $U = \{a,b\}$, $V = \{1,2\}$, and p is some proposition that takes two inputs, one from U and one from V. Then

$$(\forall x)_U ((\exists y)_V\, p(x,y))$$

("For all x in U, there exists a y in V such that $p(x,y)$ is true") means

$$((\exists y)_V\, p(a,y)) \wedge ((\exists y)_V\, p(b,y))$$

which means

$$(p(a,1) \vee p(a,2)) \wedge (p(b,1) \vee p(b,2)).$$

Likewise,

$$(\exists x)_U ((\forall y)_V\, p(x,y))$$

("There exists an x in U such that for all y in V, $p(x,y)$ is true") means

$$((\forall y)_V\, p(a,y)) \text{ or } ((\forall y)_V\, p(b,y))$$

which means

$$(p(a,1) \wedge p(a,2)) \vee (p(b,1) \wedge p(b,2)).$$

Is the proposition

(1) $(\exists y)_{\mathbb{Z}} \, (\forall x)_{\mathbb{Z}} \, (x+y \text{ is even})$

true or false?

..?..

False. (What if x is y+1?)

What about the proposition

(2) $(\forall x)_{\mathbb{Z}} \, (\exists y)_{\mathbb{Z}} \, (x+y \text{ is even})$?

..?..

True. (Just take y=x.)

But (2) is just (1) with the quantifiers reversed.

Moral: The order of quantifiers matters!

Group work: 3.8.4(ab) (6 minutes)

(http://jamespropp.org/2190/3.8.4.png)

Let the universe of discourse, U, be the set of all people, and let $M(x, y)$ be "x is the mother of y."

Which of the following is a true statement? Translate it into English.

(a) $(\exists x)_U((\forall y)_U(M(x, y)))$

(b) $(\forall y)_U((\exists x)_U(M(x, y)))$

[redacted]

..?..

Assertion (a) says that there's a single person who is EVERYONE'S mother (including her own!); this is false.

Assertion (b) says that everyone has a mother, which is true.

(Note: Neither of the two assertions asserts that every mother has a child.)

Recall:

$(\forall x)_U\, p(x)$ is equivalent to $T_p = U$.

$(\exists x)_U\, p(x)$ is equivalent to $T_p \neq \emptyset$.

In the case where U is itself empty, the former holds (T_p and U are both the empty set).

If U is the empty set then $(\forall x)_U\, p(x)$ is (vacuously) true no matter what p is, and $(\exists x)_U\, p(x)$ is false no matter what p is.

Example: U is the set of unicorns (the empty set).
Let $p(x)$ be the proposition "x is pink".
Then $(\forall x)_U\, p(x)$ ("All unicorns are pink") is true, while

$(\exists x)_U\, \neg p(x)$ ("There exists a non-pink unicorn") is false.

Compare with the vacuously true implication "If Mary is a unicorn, then Mary is pink."

Group work: 3.8.7(a) (7 minutes)

(see http://jamespropp.org/2190/3.8.7.png)

What do the following propositions say, where U is the power set of $\{1, 2, \ldots, 9\}$? Which of these propositions are true?

(a) $(\forall A)_U |A| \neq |A^c|$.

[redacted]

Note: A is an element of the universe U, which is to say, it is a subset of $\{1,2,3,4,5,6,7,8,9\}$.

Don't confuse *numbers* with *sets of numbers* or confuse *sets of numbers* with *sets of sets of numbers*. If a is a number, $|a|$ is its absolute value; if A is a set, $|A|$ is its cardinality.

Also note that in this problem, A^c means $\{1,2,\ldots,9\} - A$; that is, the set of elements of $\{1,2,\ldots,9\}$ that aren't in A. That is, for purposes of interpreting the symbol A^c, the "universe" is $\{1,2,\ldots,9\}$.

..?..

By the Inclusion-Exclusion Formula,

$$|A \cup B| = |A| + |B| - |A \cap B|,$$

so taking $B = A^c$ we get

$$|A \cup A^c| = |A| + |A^c| - |A \cap A^c|;$$

but $A \cup A^c$ is $\{1,2,3,\ldots,9\}$, with cardinality 9, while $A \cap A^c$ is the empty set, with cardinality 0, so the inclusion-exclusion formula becomes

$$9 = |A| + |A^c| - 0,$$

or just $|A| + |A^c| = 9$.

If $|A| = |A^c|$, then we have $2|A| = |A| + |A| = 9$, or $|A| = 9/2$, which is impossible. This contradiction shows that $|A| \neq |A^c|$. Since we have shown that, for all elements A of the power set of $\{1,2,\ldots,9\}$, $|A| \neq |A^c|$, we have proved that $(\forall A)\ |A| \neq |A^c|$. So the assertion is TRUE.

(A variant proof argues that when $|A|$ is odd $|A^c|$ is even and vice versa, so they can't be equal.)

Group work: 3.8.7(b) (7 minutes)

What do the following propositions say, where U is the power set of $\{1, 2, \ldots, 9\}$? Which of these propositions are true?

[redacted]

(b) $(\exists A)_U(\exists B)_U(|A| = 5, |B| = 5, \text{ and } A \cap B = \emptyset)$.

[redacted]

..?..

If A,B are subsets of $\{1,2,\ldots,9\}$ of cardinality 5 with $A \cap B = \varnothing$, we have $|A \cup B| = |A| + |B| - |A \cap B| = 5 + 5 - 0 = 10$. But since $A \cup B$ is a subset of $\{1,2,\ldots,9\}$, $|A \cup B| \leq 9$. This contradiction shows that no such sets A,B exist. So the assertion is FALSE.

The first exam will cover sections 1.1 through 3.9