[Get 75% of students to turn on cameras]

Section 3.9: A Review of Methods of Proof

- Write simple direct and indirect proofs. (For example, problems 1, 2, and 5 from this section.)

The first exam will cover sections 1.1 through 3.9.

Direct proof: To prove $P \Rightarrow Q$, assume P and prove Q.

Indirect proof (also called proof by contradiction): To prove $P \Rightarrow Q$, assume P and $\neg Q$ and derive a contradiction.

An example of direct proof is the following proof that the sum of an odd number and an even number must be odd.

(Here an odd positive integer is defined as a positive integer that leaves remainder 1 when you divide it by 2. That is, "n is odd" is equivalent to "$(\exists k)_{\mathbb{Z}}\ n=2k+1$" just as "$n$ is even" is equivalent to "$(\exists k)_{\mathbb{Z}}\ n=2k$".)

Claim: If m is odd and n is even, $m+n$ is odd.
Proof: Since n is even, we can write $n=2k$ for some $k \in \mathbb{Z}$, and since m is odd, we can write $m=2j+1$ for some $j \in \mathbb{Z}$. (Don't use the same letter for j and k; they can be unequal!) Then $m+n = (2k) + (2j+1) = 2j + 2k + 1 = 2(j+k)+1$, so $m+n$ is odd.

Group work (10 minutes): Prove that the sum of two odd numbers must be even.

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Theorem: The sum of two odd integers is an even integer.

Proof: Let m and n be odd integers. Since m and n are odd, we can write $m=2j+1$ and $n=2k+1$ for integers j,k. But then $m+n = (2j+1) + (2k+1) = 2(j+k+1)$, so $m+n$ is even.

Group work (5 minutes): Prove that the product of two odd numbers must be odd.

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Theorem: The product of two odd integers is an odd integer.

Proof: Let m and n be odd integers. Since m and n are odd, we can write $m=2j+1$ and $n=2k+1$ for integers j,k. But then $mn = (2j+1)(2k+1) = 4jk+2j+2k+1 = 2(2jk+j+k)+1$, so $m+n$ is odd.

An example of indirect proof is Exercise 3.9.3:

Write out a complete proof that $\sqrt{2}$ is irrational.

(A number is irrational if it cannot be written as the quotient p/q with $p,q \in \mathbb{Z}$ and $q \neq 0$.)

Example 3.9.3 **$\sqrt{2}$ is irrational.** Our final example will be an outline of the proof that the square root of 2 is irrational (not an element of $\mathbb{Q}$). This is an example of the theorem that does not appear to be in the standard $P \Rightarrow C$ form. One way to rephrase the theorem is: If x is a rational number, then $x^2 \neq 2$. A direct proof of this theorem would require that we verify that the square of every rational number is not equal to 2. There is no convenient way of doing this, so we must turn to the indirect method of proof. In such a proof, we assume that x is a rational number and that $x^2 = 2$. This will lead to a contradiction. In order to reach this contradiction, we need to use the following facts:

- A rational number is a quotient of two integers.
- Every fraction can be reduced to lowest terms, so that the numerator and denominator have no common factor greater than 1.
- If n is an integer, n^2 is even if and only if n is even.

Group work: 10 minutes

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Theorem: The square root of 2 is irrational.

Proof: By contradiction. Suppose sqrt(2) were rational; say sqrt(2) = p/q for positive integers p,q.

Suppose also that p and q have no common factors > 1.

Since sqrt(2) = p/q, we get $2 = p^2 / q^2$, so

$$2\,q^2 = p^2.$$

Since $2\,q^2$ is even, p^2 must be even, so p must be even; say $p = 2k$.

Rewriting $2\,q^2 = p^2$ in terms of k gives

$$2\,q^2 = (2k)^2 = 4\,k^2;$$

dividing both sides of $2\,q^2 = 4\,k^2$ by 2 gives

$$q^2 = 2\,k^2.$$

So q^2 is even, so q must be even.

So p and q are both even: they have 2 as a common factor.

But this contradicts our assumption that p/q was in reduced form.

This contradiction shows that our assumption (the rationality of the square root of 2) is untenable.

Therefore the square root of 2 is irrational.

Another example of indirect proof is Exercise 3.9.5:

Prove that if x and y are real numbers such that $x + y \leq 1$, then $x \leq \frac{1}{2}$ or $y \leq \frac{1}{2}$.

Group work: 10 minutes

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Theorem: If x and y are real numbers such that $x+y \leq 1$, then $x \leq ½$ or $y \leq ½$.

Proof: By contradiction. Assume $x+y \leq 1$, and suppose that "$x \leq ½$ or $y \leq ½$" is false; that is, suppose $x > ½$ and $y > ½$. Adding these last two inequalities, we get $x + y > 1$, contradicting $x+y \leq 1$.

(Recall De Morgan's Law $\neg(p \vee q) \Leftrightarrow \neg p \wedge \neg q$ from section 3.4.)

Here's a direct proof:

Theorem: If x and y are real numbers such that $x+y \le 1$, then $x \le ½$ or $y \le ½$.

Proof:

We use the law of trichotomy, which says that exactly one of the relations $x = y$, $x < y$, $x > y$ holds.

Case 1: Suppose $x = y$. Then $x+y \le 1$ beomes $2x \le 1$, and dividing both sides by 2 gives $x \le ½$.

Case 2: Suppose $x < y$. Then adding x to both sides gives us $2x < x+y$, and $x+y \le 1$, so $2x < 1$. Now divide both sides by 2 as in case 1.

Case 3: Just like case 2, but with x and y switching roles.

Other questions on 3.9?