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Section 5.3: Laws of Matrix Algebra / Section 5.4: Matrix Oddities

- Be familiar with the laws of matrix algebra and "matrix oddities" (Observation 5.4.2). (Memorizing the laws is not required.)
- Be familiar with the similarities and differences between elementary algebra and matrix algebra (e.g. section 5.4.1).
- Be able to produce examples of matrices which satisfy the "matrix oddities."

Note: the plural of "matrix" is "matrices", not "matrixes".

Also: the singular of "matrices" is "matrix", not "matrice".

See multiply.pdf.

Group work: Find a formula for the inverse of ABC (valid when A, B, and C are invertible n-by-n matrices), analogous to the formula $(AB)^{-1} = B^{-1}A^{-1}$. (8 minutes)

..?..

Answer: $C^{-1}B^{-1}A^{-1}$.

Method 1: Guess-and-check. (I.e., guess that that's the answer and then check that both $(ABC)(C^{-1}B^{-1}A^{-1})$ and $(C^{-1}B^{-1}A^{-1})(ABC)$ simplify to I.) ‘

Method 2: Isolate-and-solve. Start with $(ABC)^{-1}ABC = I$ and solve for $(ABC)^{-1}$ by multiplying both sides of the equation on the right by C^{-1}, B^{-1}, and A^{-1} in succession.

Method 3: "Induction": $(ABC)^{-1} = ((AB)C)^{-1} = C^{-1}(AB)^{-1} = C^{-1}B^{-1}A^{-1}$.

1. What's an example of two matrices A,B such that AB and BA aren't equal?

..?..

In one kind of example, AB is defined but BA isn't, or vice versa. For instance

$$\begin{pmatrix} 1 \\ 2 \end{pmatrix} \begin{pmatrix} 4 & 5 & 6 \end{pmatrix}$$

equals

$$\begin{pmatrix} 4 & 5 & 6 \\ 8 & 10 & 12 \end{pmatrix}$$

but

$$\begin{pmatrix} 4 & 5 & 6 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

is undefined.

We can also have examples where AB and BA are both defined but don't have the same size (number of rows, number of columns):

$$\begin{pmatrix} 1 \\ 2 \end{pmatrix} \begin{pmatrix} 4 & 5 \end{pmatrix}$$

equals the 2-by-2 matrix

$$\begin{pmatrix} 4 & 5 \\ 8 & 10 \end{pmatrix}$$

but

$$\begin{pmatrix} 4 & 5 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

equals the 1-by-1 matrix

$$\begin{pmatrix} 14 \end{pmatrix}.$$

What about 2-by-2 matrices for which AB and BA are both defined and of the same size, but aren't equal to each other?

..?.. (3 minutes)

One example of such a pair of matrices is the pair

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \text{ and } \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

Multiply them one way, you get

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix};$$

multiply them the other way, you get

$$\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

2. What are some examples of square matrices A,B such that A and B are non-zero (that is, are not equal to the all-0's matrix) yet AB equals the all-0's matrix)?

..?.. (3 minutes)

$$\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & -1 \end{pmatrix},$$

$$\begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix} \begin{pmatrix} -1 & -1 \\ 1 & 1 \end{pmatrix}, \text{ and}$$

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}.$$

3. What are examples of a single non-zero matrix A such that A^2 equals the 0 matrix?

..?.. (3 minutes)

$$\begin{pmatrix} -1 & -1 \\ 1 & 1 \end{pmatrix} \text{ and } \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

4. Find a matrix A such that A is neither the 0-matrix (consisting of 0's) nor the identity matrix I (consisting of 1's on the main diagonal and 0's everywhere else), yet $A^2 = A$:

..?.. (3 minutes)

$$\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \text{ and } \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \text{ and } \begin{pmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix}.$$

5. Lastly, find a matrix A such that A is equal to neither I nor $-I$, yet $A^2 = I$:

..?..

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \text{ and } \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Group work: Exercise 5.4.3(a): Suppose A is a square matrix (not necessarily 2-by-2). Prove or disprove the implication "$A^2 = A$ and $\det A \neq 0 \Rightarrow A = I$".

..?.. (5 min)

"If $A^2 = A$ and $\det A \neq 0$, then $A = I$" is **true**.
For, recall that when $\det A \neq 0$, A^{-1} exists.
So $A^2 = A$ implies $A^2 A^{-1} = AA^{-1}$, which reduces to $A = I$.

Group work: Exercise 5.4.3(b): Suppose A is a square matrix (not necessarily 2-by-2). Prove or disprove the implication "$A^2 = I$ and $\det A \neq 0 \Rightarrow A = I$ or $A = -I$".

..?.. (5 min)

"If $A^2 = I$ and $\det A \neq 0$, then $A = I$ or $A = -I$" is **false**.

For instance, take A to be

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \text{ or } \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

as in Example 5 from the discussion of matrix oddities.

Other questions on sections 5.3 and 5.4?