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A predicate (or proposition) p over a set U is a function from U to {True, False}, and is not to be confused with T_p, which is a subset of U.

Another name for a predicate over U is a *unary predicate* over U.

We also have *binary predicates*, which are functions from $U \times U$ to {True, False}.

For instance, "n is prime" is a unary predicate over $\mathbb{P}$ (the universe of positive integers), while "m and n are relatively prime" (i.e., have no common factor other than 1) is a binary predicate over $\mathbb{P}$.

If $p(x,y)$ is a binary predicate over U, its truth set is the set $T_p = \{(x,y): p(x,y) = \text{True}\}$, a subset of $U \times U$.

Seems sensible, right?

But then some mathematicians decided that a binary relation like "*m* is relatively prime to *n*" should not just be *represented* by its truth set T_p, but *defined* as that set! That is, even though we *think* of a binary relation like "<" as a function that takes two numbers as inputs and spits out "True" or "False" as its output, we *define* $<$ as a set of ordered pairs: (1,2), (3,5), (–3,–1), etc. (but not (4,4) or (5,3) or …).

There are historical reasons for this convention, having to do with attempts to put math on a rigorous foundation. Anyway, it's the approach many authors take (including Doerr and Levasseur), so we'll accept it.

Section 6.1: Basic Definitions

- Know basic terminology: relation from A into B, relation on set A
- Understand the *divides* relation on $\mathbb{Z}$.
- Visualize a relation using a graph (e.g. Figure 6.1.6).
- Compute the composition of relations.

Questions about section 6.1?

If S is the set $\{1,2,3\}$, the binary relation "<" (which returns the value True or False for every expression of the form "$x < y$" with x,y in S) can be uniquely specified by the pairs (1,2), (1,3), and (2,3), since these are precisely the pairs (x,y) in $S \times S$ satisfying $x < y$.

We therefore **define** "<" to be $\{(1,2),(1,3),(2,3)\}$ in $S \times S$.

More generally, a binary relation r on a set S is defined as a subset of $S \times S$.

For $r \subseteq S \times S$, we write "$a\ r\ b$" (with a,b in S) to mean the assertion that $(a,b) \in r$.

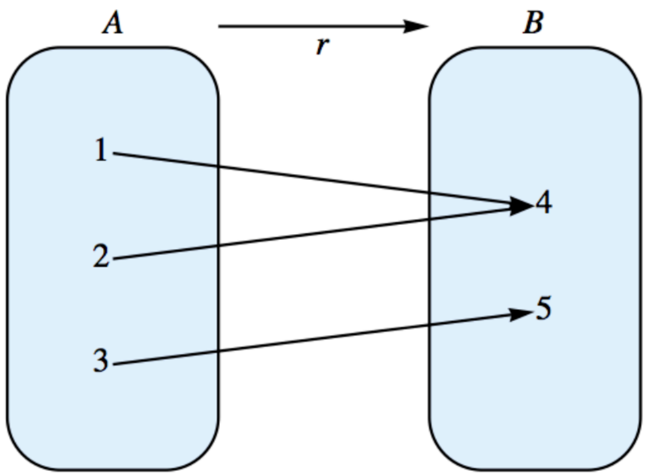
More generally, if we have a relation r from one set A to another set B, that is, if we have $r \subseteq A \times B$, we write "$a\ r\ b$" (with a in A and b in B) to mean the assertion that $(a,b) \in r$.

Note (re Example 6.1.3): We say "a divides evenly into b" if a and b are integers such that $a \neq 0$ and b/a is an integer. So if $S = \{1,2,3,4\}$, the relation "divides", as a subset of $S \times S$, is

..?..

$\{(1,1), (1,2), (1,3), (1,4), (2,2), (2,4), (3,3), (4,4)\}$.

We can represent a relation from A to B by a directed graph with arrows from A to B:

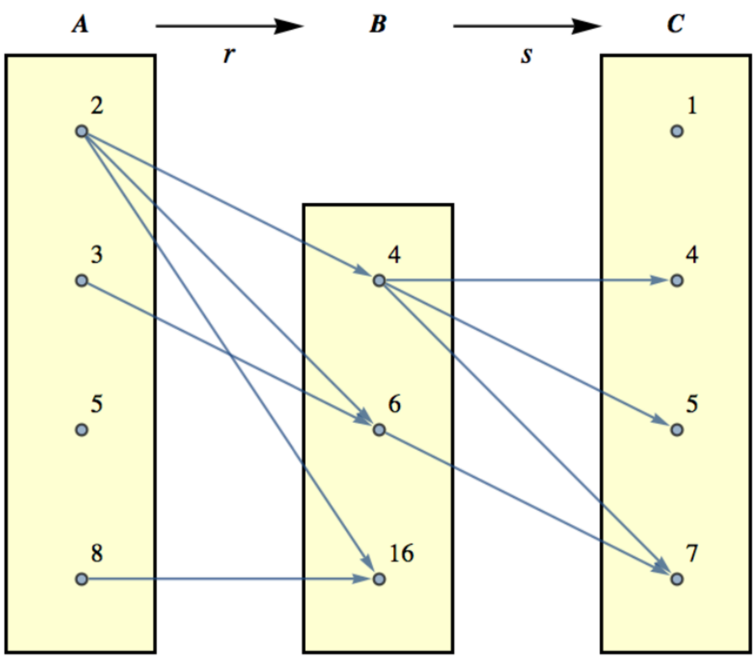


In this picture, r, as a subset of $A \times B$, is the set

..?..

$\{(1,4), (2,4), (3,5)\}$.

There's a natural way to compose two relations:



If r is a relation from A to B and s is a relation from B to C, then rs is the relation from A to C made up of all pairs (a,c) for which there exists some b in B such that $a\ r\ b$ and $b\ s\ c$.

If $r = \{(2,4), (2,6), (2,16), (3,6), (8,16)\}$
and $s = \{(4,4), (4,5), (4,7), (6,7)\}$,
then $rs = \{(2,4), (2,5), (2,7), (3,7)\}$.

Example: a school in which each child has one or more guardians, each of whom has one or more phone numbers. A = set of children, B = set of parents, C = set of parental phone numbers. (Note that there may be two paths of length 2 from some child to some phone number, if two of the child's parents share a phone number.)

Questions on section 6.1?

Group work: 6.1.1 (6 minutes): available at

http://jamespropp.org/2190/6.1.1.pdf

1. For each of the following relations r defined on $\mathbb{P}$, determine which of the given ordered pairs belong to r

(a) xry iff $x|y$; (2, 3), (2, 4), (2, 8), (2, 17)

(b) xry iff $x \leq y$; (2, 3), (3, 2), (2, 4), (5, 8)

(c) xry iff $y = x^2$; (1,1), (2, 3), (2, 4), (2, 6)

Let $S = \{1,2\}$. Let r and s be relations from S to S given by $r = \{(1,1)\}$ and $s = \{(1,2)\}$. What is rs? What is sr? Are they equal?

..?..

Answer: $rs = \{(1,2)\}$; $sr = \{\}$ = the empty set.

Compare: $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ but $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

Note that $\{\}$ is a relation on S; it's just not a very interesting one! It's the relation on S that's always false. Ditto for $S \times S$; it's the relation on S that's always true.

Other questions on section 6.1?

Section 6.2: Graphs of Relations on a Set

- Draw digraphs for relations.
- Given the digraph for a relation, write down the set of ordered pairs for the relation.

Questions on section 6.2?

When we draw a relation on a set (that is, from a set to itself), we can draw two copies of the set (as we did above), or we can draw just one.

Group work: 6.2.1 (6 minutes): available at http://jamespropp.org/2190/6.2.1.pdf

1. Let $A = \{1, 2, 3, 4\}$, and let r be the relation $\leq$ on A. Draw a digraph for r.

(We'll learn shortly that *r* is an example of a *partial ordering*.)

Group work: 6.2.3 (6 minutes) (available at

http://jamespropp.org/2190/6.2.3.pdf

3. Let $A = \{1, 2, 3, 4, 5\}$. Define t on A by atb if and only if $b - a$ is even. Draw a digraph for t.

(We'll learn shortly that t is an example of an *equivalence relation*.)

Other questions on section 6.2?