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Section 6.3: Properties of Relations

- Know the definitions of the following properties: Reflexive, symmetric, antisymmetric, transitive, partial ordering, equivalence relation
- Determine if a given relation has a certain property or not.
- Draw the Hasse diagram for a partially ordered set.

If A is a set and $\preceq$ is a binary relation on A (so that for all x,y in A, "$x \preceq y$" has the value True or the value False), we say $(A, \preceq)$ is a poset (short for **p**artially **o**rdered **set**) if the relation $\preceq$ is reflexive, antisymmetric, and transitive.

Reflexive: $(\forall\ x \in A)\ x \preceq x$

Antisymmetric: $(\forall\ x,y \in A)\ x \preceq y$ and $y \preceq x \ \Rightarrow\ x = y$

Transitive: $(\forall\ x,y,z \in A)\ x \preceq y$ and $y \preceq z \ \Rightarrow\ x \preceq z$.

I'll pronounce $x \preceq y$ as "x is dominated by y" or as "y dominates x".

Note: Doerr and Levassuer define "$\preceq$ is antisymmetric" to mean "$(x \preceq y$ and $\neg (x = y)) \Rightarrow \neg (y \preceq x)$". But we saw in Lecture #7 that "$(p$ and $\neg r) \Rightarrow \neg q$" is logically equivalent to "$(p$ and $q) \Rightarrow r$", so their definition is equivalent to mine.

Examples of posets:

A is $\mathbb{R}$ (the set of real numbers), $\preceq$ is $\leq$

A is $\mathbb{P}$ (the set of positive integers), $\preceq$ is $\mid$ (divides)

(where "$a \mid b$" means "b/a is an integer")

A is the power set of some set S, $\preceq$ is $\subseteq$.

Other common posets are subposets of these: for instance, D_n (the finite subset of $\mathbb{P}$ containing only the divisors of n) with the relation $\mid$ (divides).

In a poset we can use the given relation $\preceq$ to define three other relations:

"$x \prec y$" means "$x \preceq y$ but $x \neq y$"

"$x \succeq y$" means "$y \preceq x$"

"$x \succ y$" means "$y \prec x$"

I'll pronounce $x \preceq y$ as "x is dominated by y",

$x \prec y$ as "x is strictly dominated by y",

$x \succeq y$ as "x dominates y",

and $x \succeq y$ as "x strictly dominates y".

Doerr and Levasseur write "*r*" instead of "$\preceq$", which I find confusing; it's hard to keep in mind that *x* and *y* are *elements* of *A* and *r* is a *relation* on *A*. But the name *r* makes sense when we think of a relation not as a function from $A \times A$ to {True, False} but as a subset of $A \times A$.)

Sometimes people write "$\leq$" instead of "$\preceq$". So keep in mind that "$\leq$" is sometimes used to denote a partial ordering on some set *A* rather than the specific ordering relation on $\mathbb{R}$ you studied in grade school.)

Questions on poset.pdf?

If x and z are elements of a poset L, we say z covers x iff

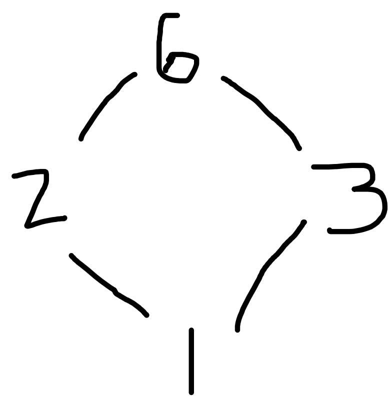
(1) $x \prec z$ and

(2) there does not exist any y such that $x \prec y$ and $y \prec z$.

Example: In $\mathbb{Z}$, 6 covers 5, but in $\mathbb{R}$, 6 does not, because (for instance) $6 > 5.5 > 5$.

In the Hasse diagram of a poset, we draw an edge joining x and y iff y covers x, and we draw y above x. Then to decide whether $x \preceq y$ in the poset, we just have to determine "Is there an upward path from x to y?" (Here staying put counts as an upward path of length 0.)

Example: In $D_6 = \{1,2,3,6\}$, the relation "divides", viewed as a set of ordered pairs, is {(1,1), (1,2), (1,3), (1,6), (2,2), (2,6), (3,3), (3,6), (6,6)}, and the Hasse diagram is



In the poset D_6, 6 *dominates* what elements?

..?..

1, 2, 3, and 6.

6 *strictly dominates* what elements?

..?..

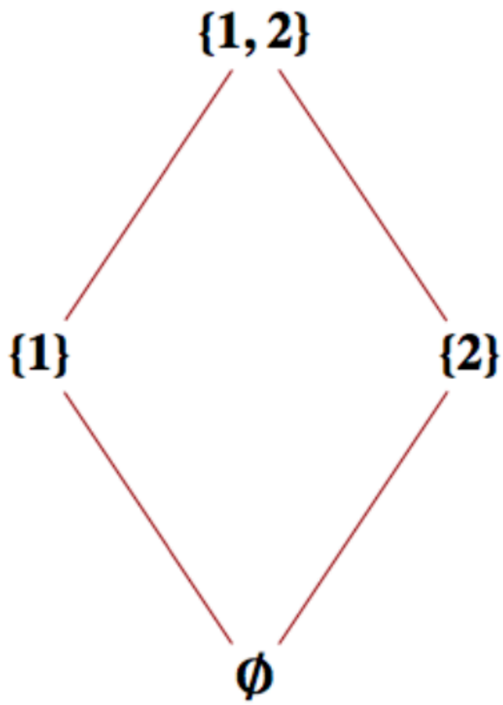
1, 2, and 3.

6 *covers* what elements?

..?..

2 and 3.

Note that the Hasse diagram for D_6 and the Hasse diagram for Pow({1,2}) shown in Figure 6.3.6 are the same picture: only the labels have changed.

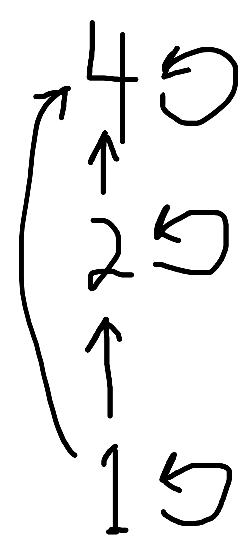


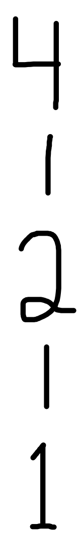
In Discrete Structures II we'll learn to say that these partially ordered sets are *isomorphic*.

Group work (5 minutes): determine the poset D_4 as a set of ordered pairs and as a directed graph, and draw its Hasse diagram.

..?..

{(1,1), (1,2), (1,4), (2,2), (2,4), (4,4)}





If A is a set and $\equiv$ is a binary relation on A (so that for all x,y in A, "$x \equiv y$" has the value True or the value False), we say $\equiv$ is an equivalence relation on A if the relation $\equiv$ is reflexive, symmetric, and transitive.

Reflexive: $(\forall\ x \in A)\ x \equiv x$

Symmetric: $(\forall\ x,y \in A)\ x \equiv y \ \Rightarrow\ y \equiv x$

Transitive: $(\forall\ x,y,z \in A)\ x \equiv y$ and $y \equiv z \ \Rightarrow\ x \equiv z$.

Examples of equivalence relations:

A is $\mathbb{Z}$, $\equiv$ is congruence mod n (for some positive integer n)

A is the set of triangles in the plane, $\equiv$ is congruence

A is the set of triangles in the plane, $\equiv$ is similarity

Whenever ≡ is an equivalence relation on A, the set A is partitioned into equivalence classes, where the equivalence class of x is the set of all y in A (including x itself) such that $x \equiv y$.

(Conversely, given a partition of A into nonempty subsets $B_1, B_2, \ldots$, we can define an equivalence relation ≡ such that $x,y \in A$ satisfy $x \equiv y$ iff x and y belong to the same B_i.)

When A is $\mathbb{Z}$ and ≡ is congruence mod n, the equivalence classes are the arithmetic progressions of difference n, also called congruence classes mod n.

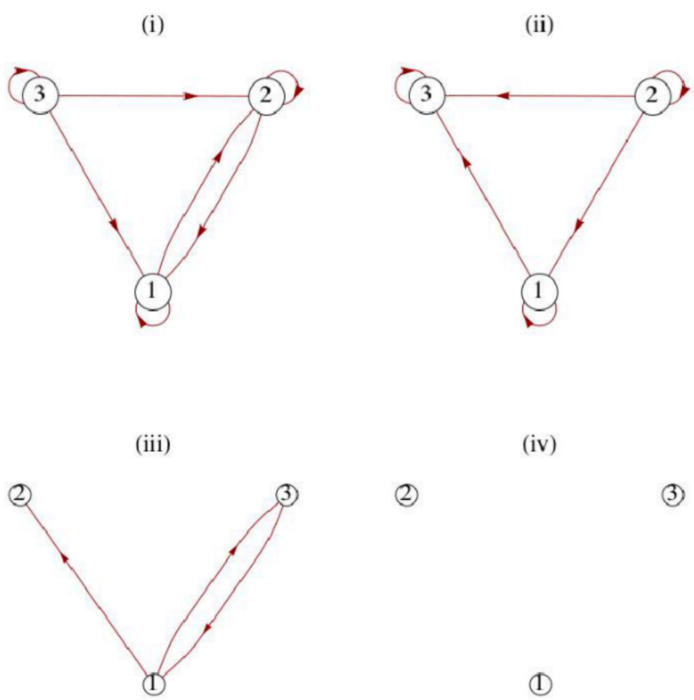
Group work: 6.3.3 (first four; 6 minutes): available at

http://jamespropp.org/2190/6.3.3.a.pdf

3. Consider the relations defined by the digraphs in Figure 6.3.17.

(a) Determine whether the given relations are reflexive, symmetric, antisymmetric, or transitive. Try to develop procedures for determining the validity of these properties from the graphs,

(b) Which of the graphs are of equivalence relations or of partial orderings?



Group work: 6.3.3 (last three; 6 minutes): available at

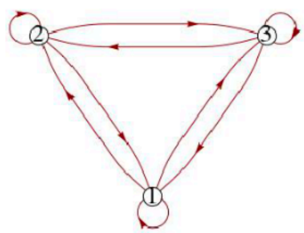
http://jamespropp.org/2190/6.3.3.b.pdf

3. Consider the relations defined by the digraphs in Figure 6.3.17.

(a) Determine whether the given relations are reflexive, symmetric, antisymmetric, or transitive. Try to develop procedures for determining the validity of these properties from the graphs,

(b) Which of the graphs are of equivalence relations or of partial orderings?

(v)



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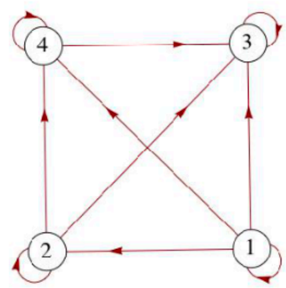
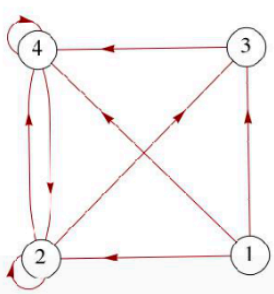


(vii)



Discuss 6.3.7, highlighting notion of equivalence classes

7. Equivalence Classes. Let $A = \{0, 1, 2, 3\}$ and let

$$r = \{(0,0), (1,1), (2,2), (3,3), (1,2), (2,1), (3,2), (2,3), (3,1), (1,3)\}$$

(a) Verify that r is an equivalence relation on A.

(b) Let $a \in A$ and define $c(a) = \{b \in A \mid arb\}$. $c(a)$ is called the **equivalence class of** a **under** r. Find $c(a)$ for each element $a \in A$.

(c) Show that $\{c(a) \mid a \in A\}$ forms a partition of A for this set A.

(d) Let r be an equivalence relation on an arbitrary set A. Prove that the set of all equivalence classes under r constitutes a partition of A.

The partition of A given by r is $\{ \{0\} , \{1,2,3\} \}$.

Other questions on section 6.3?