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Read my email about the final exam.

Section 7.1: Definition and Notation

- Know basic terminology: function, domain, codomain, image, range

[Have students define/discuss the words "function", "domain", "codomain", "mapping", "map", "transformation", "range".]

What's the difference between "codomain" and "range"?
..?..
The codomain consists of all the values that the function is "allowed" to have; the range consists of all the values that the function *actually achieves*.
Classic example: For the function $f(x) = x^2$, the codomain is $\mathbb{R}$ but the range is just the set of nonnegative reals.

When f is a function from set A to set B, with a an element of A and with b an element of B satisfying $f(a) = b$, we often say that f "sends" a to b, or "carries" a to b, or "maps" a to b. (E.g., "The squaring functions sends –2 to +4.")

[Give three representations for the function f: $\{1,2\} \rightarrow \{1,2,3,4\}$ given by $f(x) = x^2$ (one a set of ordered pairs, one a graph in the first quadrant, and one a directed graph in the style of chapter 6).]

A relation *r* from *A* to *B* is a function if and only if for every *a* in *A*, there is exactly one *b* in *B* such that *a r b* is true, i.e., there is exactly one *b* in *B* such that (*a*,*b*) belongs to our list of ordered pairs.

More pictorially: the relation *r* is a function if and only if, in the graph of *r*, each element of *A* has exactly one arrow coming out of it.

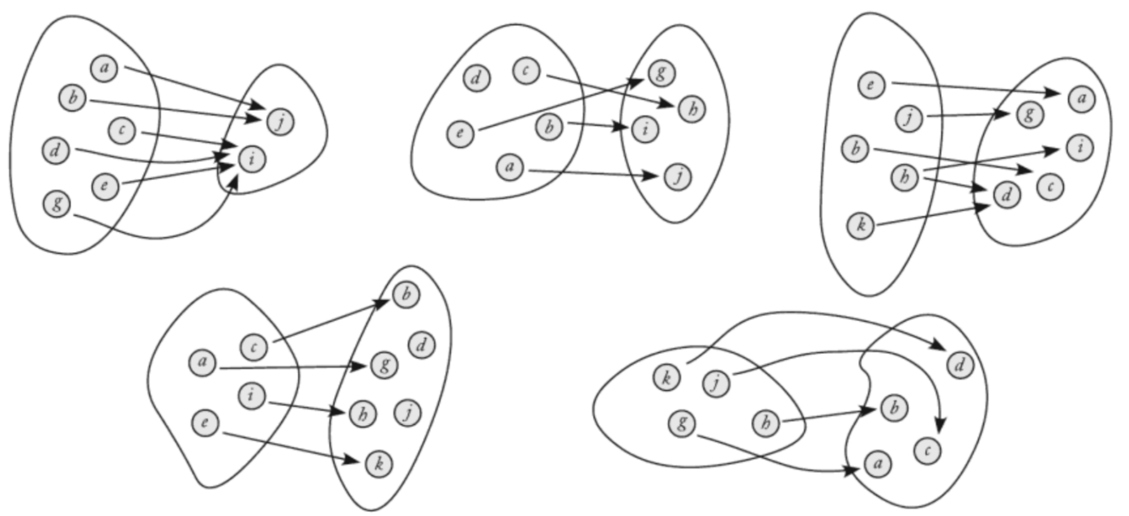

Figure 3.1. Exactly three of these are functions. Which three?

(figure taken from sarah-marie belcastro's book "Discrete Mathematics with Ducks")

Which three of the five relations are functions?

[Relate this to calculus: the vertical line test]

Questions on section 7.1?

Group work: 7.1.1, 7.1.3 (10 minutes)

Exercise 1. Let $A = \{1,2,3,4\}$ and $B = \{a,b,c,d\}$. Determine which of the following are functions. Explain.

(a) $f \subseteq A \times B$, where $f = \{(1,a),(2,b),(3,c),(4,d)\}$.

(b) $g \subseteq A \times B$, where $g = \{(1,a),(2,a),(3,b),(4,d)\}$.

(c) $h \subseteq A \times B$, where $h = \{(1,a),(2,b),(3,c)\}$.

(d) $k \subseteq A \times B$, where $k = \{(1,a),(2,b),(2,c),(3,a),(4,a)\}$.

(e) $L \subseteq A \times A$, where $L = \{(1,1),(2,1),(3,1),(4,1)\}$.

Exercise 3. Find the ranges of each of the relations in Exercise 1 that are functions.

Section 7.2: Properties of Functions

- Know the definitions for the following terminology: injective, surjective, bijective.
- Identify when a function is injective, surjective, and/or bijective.

[Have students define/discuss the words “injective”, “surjective”, “bijective”.]

Of the three functions shown in the “Discrete Mathematics with Ducks” picture (ignore the two non-functions!), which are injective? surjective? bijective?

..?..

You've seen the quantifiers ∀ and ∃. Another common quantifier is ∃!, meaning "there exists exactly one", as in the true assertion $(\exists! x)_{\mathbb{R}}\ x^2 = 0$ ("There is only one real number x such that x^2 equals 0"). Note that $(\exists! x)_{\mathbb{R}}\ x^2 = 1$ is false (since $x = 1$ and $x = -1$ are both solutions).

Some people use $\exists^{\leq 1}$ to mean "there exists at most one", as in the true assertions $(\exists^{\leq 1} x)_{\mathbb{R}}\ x^2 = 0$ ("There is at most one real number x for which $x^2 = 0$") and $(\exists^{\leq 1} x)_{\mathbb{R}}\ x^2 = -1$ ("There is at most one real number x satisfying $x^2 = -1$"). Note that $(\exists^{\leq 1} x)_{\mathbb{R}}\ x^2 = 1$ is false.

Why don't people use the symbol $\exists^{\geq 1}$?

..?..

Because plain-old ∃ already means the same thing.

Which of the following assertions says the relation r from X to Y is actually a function from X to Y?

(a): $(\forall x)_X\,(\exists! y)_Y\ (x\ r\ y)$

(b): $(\forall y)_Y\,(\exists! x)_X\ (x\ r\ y)$

[Chat storm]

..?..

Assertion (a) says that r is a function (or, if you prefer, that there exists a function f such that $x\ r\ y$ if and only if $y = f(x)$).

Assertion (b) says something different (more on this soon).

Remember the time-order of quantifiers:

"$(\forall x)_X\ (\exists y)_Y$ …" means "For all x in X, there exists some y in Y, possibly depending on x, such that…"

Whereas:

"$(\exists x)_X\ (\forall y)_Y$ …" means "There exists some special x belonging to X with the property that, for every y, without exception, …"

Suppose now that we already know r is a function (so we can use function notation "$y = f(x)$" instead of "$x\ r\ y$"). What do the following propositions assert? [More chat storms]

(a): $(\forall x)_X\ (\exists y)_Y\ (f(x) = y)$

..?..

It doesn't tell us anything we didn't already know, since f is a function.

(b): $(\exists y)_Y\ (\forall x)_X\ (f(x) = y)$

..?..

f is a constant function.

(c): $(\forall y)_Y\ (\exists x)_X\ (f(x) = y)$

..?..

f is surjective, aka onto.

(d): $(\exists x)_X\,(\forall y)_Y\,(f(x) = y)$

..?..

The set Y has only one element.

(e): $(\forall x)_X\,(\exists! y)_Y\,(f(x) = y)$

Again, we learn nothing new about f.

(f): $(\forall y)_Y\,(\exists! x)_X\,(f(x) = y)$

..?..

f is bijective, aka one-to-one and onto.

(g): $(\forall x)_X\,(\forall x')_X\;\;(x = x' \rightarrow f(x) = f(x'))$

..?..

We learn nothing new about f.

(h): $(\forall x)_X\,(\forall x')_X\;\;(x \neq x' \rightarrow f(x) \neq f(x'))$

..?..

f is injective, aka one-to-one.

Let’s critique/fix these actual examples of confused student writing:

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"f is an injection if for every x there is a unique f(x)."
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That sounds more like what it means to say f is a function! This student has switched the roles of the domain and codomain.

And they’re using “unique” (meaning “exactly one”) when they should say “at most one”.

Correct phrasing: “f is an injection if for every y there is at most one x satisfying $f(x) = y$.”

Here's a second example:

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"f is a surjection if for every range there exists a domain for
f(n)."
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This student is confusing "range" with "codomain", and confusing *sets* with *elements* of those sets.
(If $f(x) = x^2$, we don't say "2 is a domain of f; we say "2 is an element of the domain of f".)
Correct phrasing: "f is a surjection if for every element y of the codomain there exists an element x of the domain such that $y = f(x)$."