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The final exam will be Thursday, December 16, 6:30-9:30 pm in Olney 218.

If you have exam conflicts please let me know right away!

The format will be different from the midterm.

A practice exam will be available next week.

Chapter 8 is about sequences that satisfy **recurrence relations**. Terms of a sequence may be written in function notation ("$P(1), P(2), P(3), \ldots$") or in subscript notation ("$P_1, P_2, P_3, \ldots$"); I'll regard the two as interchangeable. But please don't write $P^1, P^2, P^3, \ldots$! That means something totally different.

Section 8.1: The Many Faces of Recursion / Section 8.2: Sequences

- Know basic terminology: Sequences
- Given a recurrence relation with initial conditions, compute additional terms in the sequence.

Note that, in keeping with the never-ending feud over whether 0 should be considered "natural", the initial term of a sequence is sometimes the 1st and sometimes the 0th.

Doerr and Levasseur like to index their sequences with the index k; I prefer to use n, though I'll sometimes use k.

Examples of sequences satisfying a recurrence relation:

The sequence $P = (1, 2, 4, 8, \ldots)$ with initial term P_0, satisfying the initial condition $P_0 = 1$ and the recurrence relation

$$P_n = 2\, P_{n-1} \text{ for all } n \geq 1,$$

is given by the closed-form formula $P_n = 2^n$.

The Virahanka (aka Hemachandra, aka Fibonacci) sequence $V = (1, 2, 3, 5, 8, \ldots)$ with initial terms V_1 and V_2, satisfying the initial conditions $V_1 = 1$ and $V_2 = 2$ and the recurrence relation

$$V_n = V_{n-1} + V_{n-2} \text{ for all } n \geq 3,$$

has an exact formula that we will discuss later.

The sequence $S = (1, 2, 6, 24, 120, \ldots)$ with initial term S_1, satisfying the initial condition $S_1 = 1$ and the recurrence relation

$$S_n = n\, S_{n-1} \text{ for all } n \geq 2,$$

is given by the formula $S_n = n!$.

The sequence $T = (1, 3, 7, 15, 31, \ldots)$ with initial term T_1, satisfying the initial condition $T_1 = 1$ and the recurrence relation

$$T_n = 2\,T_{n-1} + 1 \text{ for all } n \geq 2,$$

is given by the formula $T_n = 2^n - 1$.

The sequence $Q = (1, 1+4, 1+4+9, 1+4+9+16, \ldots) = (1, 5, 14, 30, \ldots)$ with initial term Q_1, satisfying the initial condition $Q_1 = 1$ and the recurrence relation

$$Q_n = Q_{n-1} + n^2 \text{ for all } n \geq 2,$$

is given by an exact formula you'll derive later in HW #12. Note that Q_n can be described either by the formula

$$Q(n) = \sum_{j=1}^{n} j^2,\ n \geq 1$$

or by the recurrence $Q_n = Q_{n-1} + n^2$; being able to pass back and forth between the two descriptions of the sequence is in some ways the hardest part of that homework problem.

These are all examples of linear recurrences. P, V, T, and Q are sequences satisfying linear recurrence relations with constant coefficients. The first two of these four recurrences are homogeneous; the other two are inhomogeneous. Compare:

$P_n = 2\,P_{n-1}$	1st order	homogeneous
$V_n = V_{n-1} + V_{n-2}$	2nd order	homogeneous
$T_n = 2\,T_{n-1} + 1$	1st order	nonhomogeneous
$Q_n = Q_{n-1} + n^2$	1st order	nonhomogeneous

Questions on section 8.1?

There's a relation between recursion and induction.

For instance, if we define T recursively as above, then the proof that the formula for T is correct can be proved by mathematical induction:

Theorem: If $T_1 = 1$ and $T_n = 2\ T_{n-1} + 1$ for all $n \geq 2$, then $T_n = 2^n - 1$ for all $n \geq 1$.

Proof: By mathematical induction.

Base case: For $n=1$, the claim is true.

Induction step: If $T_{n-1} = 2^{n-1} - 1$, then

$T_n = 2\ T_{n-1} + 1 = 2(2^{n-1} - 1) + 1 = 2(2^{n-1}) - 2 + 1 = 2^n - 1$.

Hence the theorem is true.

Our recipes for solving recurrence relation problems can all be justified using proof by induction.

However, we won't be concerned with induction in our study of recurrence relations; I'll keep it in the background.

Discuss the “smart” and “dumb” way to write recursive code to compute Virahanka numbers (with exponential slowdown): use memoization!

Discuss the fast and slow ways to evaluate mononomials:

$x^{64} = ((((((x^2)^2)^2)^2)^2)^2$; $x^{40} = x^{32+8} = x^{32}\,x^8$.

Discuss the fast and slow ways to evaluate polynomials:

$x^4 + 2x^3 + 3x^2 + 5x + 7 = (((x+2)x+3)x+5)x+7$

Questions on section 8.2?

Group work (5 minutes): Suppose $C(0), C(1), C(2), \ldots$ is a sequence satisfying the initial conditions

$$C(0) = 7/3 \text{ and } C(1) = 6$$

and the second-order recurrence relation

$$C(k) - 5\,C(k-1) + 6\,C(k-2) = 2k-7$$

for all $k \geq 2$. Find the value of $C(2)$.

..?..

Plugging $k = 2$ into the recurrence relation, we get

$$C(2) - 5\,C(1) + 6\,C(0) = -3$$

and substituting $C(0) = 7/3$ and $C(1) = 6$ gives us

$$C(2) - 30 + 14 = -3;$$

so

$$C(2) = 30 - 14 - 3 = 13.$$

Group work (5 minutes): Suppose $B(0), B(1), B(2), \ldots$ is a sequence satisfying the initial condition

$$B(0) = 7/3$$

and the first-order recurrence relation

$$B(k) - 3\,B(k-1) = -\,2^k - 2k + 3$$

for all $k \geq 1$. Find the values of $B(1)$ and $B(2)$.

..?..

Plugging $k = 1$ into the recurrence relation we get

$$B(1) - 3\,B(0) = -\,2^1 - 2\cdot 1 + 3 = -1$$

and substituting $B(0) = 7/3$ gives us

$$B(1) - 7 = -1,$$

so $B(1) = 6$. Then plugging $k = 2$ into the recurrence relation we get

$$B(2) - 3\,B(1) = -\,2^2 - 2\cdot 2 + 3 = -\,5$$

and substituting $B(1) = 6$ gives us

$$B(2) - 18 = -\,5$$

so $B(2) = 13$.

Notice anything?

..?..

We get the same numbers!

In fact, $B(n) = C(n)$ for all $n \geq 0$.

A problem I won't assign:

Suppose $A(0), A(1), A(2), \ldots$ is a sequence satisfying the initial conditions

(no initial conditions satisfied)

and the "zeroeth-order recurrence relation"

$$A(k) = 3^{k-1} + 2^{k+1} + k$$

for all $k \geq 0$. Find the values of $A(0)$, $A(1)$, and $A(2)$.

It's the same sequence: $A(k) = B(k) = C(k)$ for all $k \geq 0$.

This sequence also satisfies the third-order recurrence relation

$$D(k) - 6\,D(k-1) + 11\,D(k-2) - 6\,D(k-3) = 2$$

for all $k \geq 3$ and the fourth-order recurrence relation

$$E(k) - 7\,E(k-1) + 17\,E(k-2) - 17\,E(k-3) + 6\,E(k-4) = 0.$$

See Table 8.3.5 in Doerr and Levasseur.

Next week we'll learn a method for solving such recurrence relations, which are also sometimes called *difference equations* (in analogy with differential equations).

Happy Thanksgiving!