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The final exam will be Thursday, December 16, 6:30-9:30 pm in Olney 218.

If you have exam conflicts please let me know right away!

The format will be different from the midterm.

I'll be posting a practice exam by the weekend. Try the practice exam before the following Thursday. Keep in mind that the actual exam may be harder and/or formatted differently.

Section 8.3: Recurrence relations

- Solve recurrence relations using Algorithm 8.3.12.

Model problem: Solve the recurrence relation

$$S(k) = 3\, S(k-1) - 2\, S(k-2)$$

with initial conditions $S(0) = 0$, $S(1) = 1$.

1. **Find and solve the characteristic equation, and write down the general form of the solution.**

Rewrite the recurrence in the form

$$S(k) - 3\,S(k-1) + 2\,S(k-2) = 0$$

The characteristic polynomial

$$a^2 \quad - \quad 3\;a \quad + \quad 2$$

factors as $(a-1)(a-2)$, so the characteristic equation

$$a^2 - 3a + 2 = 0$$

has (simple) roots $a = 1$ and $a = 2$.

Since the roots are 1 and 2 (each with multiplicity 1), the general solution is of the form $S(k) = b_1\,1^k + b_2\,2^k$.

2. **To find b_1 and b_2, plug into the initial conditions.**

Plugging $k = 0$ and $k = 1$ into $S(k) = b_1\, 1^k + b_2\, 2^k$ we get

$S(0) = b_1\, 1^0 + b_2\, 2^0 = b_1 + b_2$

$S(1) = b_1\, 1^1 + b_2\, 2^1 = b_1 + 2b_2$

Since we also have initial conditions $S(0) = 0$ and $S(1) = 1$,

we must have

$b_1 + \ \ b_2 \ = 0$

$b_1 + 2b_2 \ = 1$

which we solve to give $b_1 = -1$, $b_2 = 1$.

So $S(k) = (-1)\, 1^k + (1)\, 2^k = 2^k - 1$.

3. **Check your answer by computing the next term in two different ways.**

Use the recurrence we were given, with k=2:

$S(2) = 3\ S(1) - 2\ S(0) = 3\ (1) - 2\ (0) = 3$

Use the formula we derived (correctly, we hope), with k=2:

$S(2) = 2^2 - 1 = 4 - 1 = 3$.

☺

Group work (6 minutes):

Consider the following recurrence relation and initial conditions (Exercise 8.3.1):

$$S(k) - 10S(k-1) + 9S(k-2) = 0,\ S(0) = 3,\ S(1) = 11$$

Find and solve the characteristic equation, and write down the general form of the solution.

..?..

The characteristic equation is $a^2 - 10a + 9 = 0$, with roots $a = 9$ and $a = 1$, so there exists constants b_1 and b_2 with $S(k) = b_1\, 9^k + b_2\, 1^k = b_1\, 9^k + b_2$.

Group work (6 minutes): Given the recurrence and initial conditions

$$S(k) - 10S(k-1) + 9S(k-2) = 0,\ S(0) = 3,\ S(1) = 11$$

you showed that $S(k) = b_1\, 9^k + b_2$ for suitable constants b_1 and b_2.

To find b_1 and b_2, plug into the initial conditions.

..?..

$3 = S(0) = b_1\, 9^1 + b_2 = b_1 + b_2$

$11 = S(1) = b_1\, 9^1 + b_2 = 9\, b_1 + b_2$.

Subtracting the first equation from the second, we get $8 = 8\, b_1$, so $b_1 = 1$, and plugging that into the other equation we get $b_2 = 2$.

Hence the desired answer is $S(k) = 9^k + 2$.

Group work (6 minutes): Given the recurrence and initial conditions

$$S(k) - 10S(k-1) + 9S(k-2) = 0,\ S(0) = 3,\ S(1) = 11$$

you showed that $S(k) = 9^k + 2$.

Check your answer by computing the next term in two different ways.

..?..

After you've gotten the answer $S(k) = 9^k + 2$, you can check it by computing $S(2)$ in two different ways.

First way: Use the recurrence relation

$$S(k) = 10\ S(k-1) - 9\ S(k-2)$$

We get $S(2) = 10\ S(1) - 9\ S(0) = (10)(11) - (9)(3) = 83$.

Second way: Use the formula

$$S(k) = 9^k + 2.$$

We get $S(2) = 9^2 + 2 = 83$. ☺

What if I'd given you $S(k) - 4\,S(k-1) + 4\,S(k-2) = 0$?

The characteristic equation is $a^2 - 4a + 4 = 0$, which factors as $(a - 2)\,(a - 2) = 0$, with a double root $a = 2$.
Would the general solution be $S(k) = b_1\,2^k + b_2\,2^k$?

..?..

No; the general solution is $S(k) = b_1\,2^k + b_2\,k2^k$, i.e. $S(k) = (b_1 + b_2\,k)\,2^k$ (a linear function of k times an exponential function of k).

(c) *If there are n distinct characteristic roots, $a_1, a_2, \ldots a_n$, then the general solution of the recurrence relation is $S(k) = b_1 {a_1}^k + b_2 {a_2}^k + \cdots + b_n {a_n}^k$. If there are fewer than n characteristic roots, then at least one root is a multiple root. If a_j is a double root, then the $b_j {a_j}^k$ term is replaced with $(b_{j0} + b_{j1} k)\, a_j^k$. In general, if a_j is a root of multiplicity p, then the $b_j {a_j}^k$ term is replaced with $\left(b_{j0} + b_{j1} k + \cdots + b_{j(p-1)} k^{p-1}\right) a_j^k$.*

What if I'd given you $S(k) - 10\,S(k-1) + 9\,S(k-2) = 2^k$ with some specified values of $S(0)$ and $S(1)$?

The presence of 2^k (as opposed to 0) in the RHS makes this an inhomogeneous linear recurrence relation.

The characteristic equation is

..?..

$a^2 - 10a + 9 = 0$ (as in the first problem), with roots $a = 9$ and $a = 1$.

Our solution is of the form $S(k) = S^{(h)}(k) + S^{(p)}(k)$

where $S^{(h)}(k)$ is of the form $b_1\,9^k + b_2\,1^k$ (a solution to the homogeneous recurrence $S(k) - 10\,S(k-1) + 9\,S(k-2) = 0$)

and $S^{(p)}(k)$ is of the form $d\,2^k$ (given to us by Table 8.3.17).

Table 8.3.17 Particular solutions for given right-hand sides

Right Hand Side, $f(k)$	Form of Particular Solution, $S^{(p)}(k)$
Constant, q	Constant, d
Linear Function, $q_0 + q_1 k$	Linear Function, $d_0 + d_1 k$
m^{th} degree polynomial, $q_0 + q_1 k + \cdots + q_m k^m$	m^{th} degree polynomial, $d_0 + d_1 k + \cdots + d_m k^m$
exponential function, qa^k	exponential function, da^k

To solve for d, use the fact that $S^{(p)}(k)$ must be a solution to the original recurrence $S(k) - 10\, S(k-1) + 9\, S(k-2) = 2^k$.

So

(*) $d\, 2^k - 10\, d\, 2^{k-1} + 9\, d\, 2^{k-2} = 2^k$.

Divide both sides by 2^k:

$d - 10\, d\, (1/2) + 9\, d\, (1/4) = 1 \Rightarrow (-7/4)\, d = 1 \Rightarrow d = -4/7$.

OR: Use the fact that (*) must hold for all k and plug in some convenient value to solve for d.

k=2: $d\, 2^2 - 10\, d\, 2^1 + 9\, d\, 2^0 = 2^2$ gives $4d - 20d + 9d = 4$ which also gives $d = -4/7$.

(NOTE: If your algebra gives you a value of d that depends on k, you've made a mistake; d is a constant!)

So our solution is of the form

$$S(k) = S^{(h)}(k) + S^{(p)}(k)$$
$$= b_1\, 9^k + b_2\, 1^k - (4/7)\, 2^k.$$

How do we find b_1 and b_2 ?

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Plug into the initial conditions for $S(0)$ and $S(1)$.