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The final exam will be Thursday, December 16, 6:30-9:30 pm in Olney 218.

If you have exam conflicts please let me know right away!

The format will be different from the midterm.

I'll be posting a practice exam by the weekend. Try the practice exam before the following Thursday. Keep in mind that the actual exam may be harder and/or formatted differently.

Section 8.3: Recurrence relations

- Solve recurrence relations using Algorithm 8.3.12.

Algorithm for the homogeneous case:

1. **Find and solve the characteristic equation, and write down the general form of the solution.**

2. **To find b_1 and b_2, plug into the initial conditions.**

3. **Check your answer by computing the next term in two different ways.**

Caveat:

(c) *If there are n distinct characteristic roots, $a_1, a_2, \ldots a_n$, then the general solution of the recurrence relation is $S(k) = b_1 {a_1}^k + b_2 {a_2}^k + \cdots + b_n {a_n}^k$. If there are fewer than n characteristic roots, then at least one root is a multiple root. If a_j is a double root, then the $b_j {a_j}^k$ term is replaced with $(b_{j0} + b_{j1}k)\, a_j^k$. In general, if a_j is a root of multiplicity p, then the $b_j {a_j}^k$ term is replaced with $\left(b_{j0} + b_{j1}k + \cdots + b_{j(p-1)}k^{p-1}\right) a_j^k$.*

Example: For the Virahanka sequence 1, 1, 2, 3, 5, 8, … satisfying the recurrence relation $V(k) = V(k-1) + V(k-2)$ with initial conditions $V(1) = V(2) = 1$, the characteristic equation is $a^2 - a - 1 = 0$, with roots $\phi = (1+\mathrm{sqrt}(5))/2$ and $\phi' = (1-\mathrm{sqrt}(5))/2$. So we are guaranteed a formula of the form $V(k) = b_1 (\phi)^k + b_2 (\phi')^k$ for some constants b_1, b_2. As it happens, $V(k) = ((\phi)^k - (\phi')^k)/\mathrm{sqrt}(5)$ ("Binet's formula").

What about the sequence 1, 3, 4, 7, 11, … satisfying the same recurrence relation? Since it satisfies the same recurrence relation with the same characteristic equation, it too must have its kth term given by a formula of the form $b_1 (\phi)^k + b_2 (\phi')^k$, with different coefficients b_1, b_2.

Last time I had you solve the homogeneous recurrence relation $S(k) - 10\, S(k-1) + 9\, S(k-2) = 0$ with some specified initial terms $S(0)$ and $S(1)$.

What if I asked you to solve $S(k) - 10\, S(k-1) + 9\, S(k-2) = 2^k$ with some specified values of $S(0)$ and $S(1)$?

The presence of 2^k (as opposed to 0) in the RHS makes this an inhomogeneous linear recurrence relation.

The characteristic equation is $a^2 - 10a + 9 = 0$, with the roots $a = 9$ and $a = 1$.

Our solution is of the form $S(k) = S^{(h)}(k) + S^{(p)}(k)$

where $S^{(h)}(k)$ is of the form $b_1\, 9^k + b_2\, 1^k$ (a solution to the homogeneous recurrence $S(k) - 10\, S(k-1) + 9\, S(k-2) = 0$)

and $S^{(p)}(k)$ is of the form $d\, 2^k$ (given to us by Table 8.3.17).

Table 8.3.17 Particular solutions for given right-hand sides

Right Hand Side, $f(k)$	Form of Particular Solution, $S^{(p)}(k)$
Constant, q	Constant, d
Linear Function, $q_0 + q_1 k$	Linear Function, $d_0 + d_1 k$
m^{th} degree polynomial, $q_0 + q_1 k + \cdots + q_m k^m$	m^{th} degree polynomial, $d_0 + d_1 k + \cdots + d_m k^m$
exponential function, qa^k	exponential function, da^k

What if I'd given you $S(k) - 10\, S(k-1) + 9\, S(k-2) = 3^k$?

..?..

$S(k) = b_1\, 9^k + b_2\, 1^k + d\, 3^k$.

What if I'd given you $S(k) - 10\, S(k-1) + 9\, S(k-2) = 4^k$?

..?..

$S(k) = b_1\, 9^k + b_2\, 1^k + d\, 4^k$.

What if I'd given you $S(k) - 10\, S(k-1) + 9\, S(k-2) = 9^k$?

..?..

$S(k) = b_1\, 9^k + b_2\, 1^k + \boldsymbol{dk\, 9^k}$. See Observation 8.3.23:

Observation 8.3.23 When the base of right-hand side is equal to a characteristic root. If the right-hand side of a nonhomogeneous relation involves an exponential with base a, and a is also a characteristic root of multiplicity p, then multiply your guess at a particular solution as prescribed in Table 8.3.17 by k^p, where k is the index of the sequence.

Here's another example:

What if I'd given you $S(k) - 18\, S(k-1) + 81\, S(k-2) = 9^k$?

..?..

Then the base 9 from the inhomogeneous term 9^k is also a double root of the characteristic polynomial $a^2 - 18a + 81 = (a - 9)^2$, so its multiplicity p equals 2, so we have to multiply $d9^k$ by k^2, giving us $S(k) = b_1\, 9^k + b_2\, 1^k + \boldsymbol{dk^2\, 9^k}$.

Group work (6 minutes): Write down the general form of the solution to $S(k) - 10\, S(k-1) + 9\, S(k-2) = 1$? (Remember, for purposes of the algorithm, you should think of the RHS as 1^k; the base is a=1.)

..?..

This time our first guess is $S^{(p)}(k) = d$, but Observation 8.3.23 tells us that we have to multiply it by $k^1 = k$, giving $S^{(p)}(k) = dk$ and $S(k) = b_1\, 9^k + b_2\, 1^k + \boldsymbol{dk}$.

The trickiest aspect of this chapter is that an inhomogeneous term that's a constant, c say, must often be treated as an exponential function $c\ 1^k$.

Note that the first row of Table 8.3.17 is actually a special case of the last row: if $f(k)$ is a constant q, it's secretly of the form qa^k with a=1.

Table 8.3.17 Particular solutions for given right-hand sides

Right Hand Side, $f(k)$	Form of Particular Solution, $S^{(p)}(k)$
Constant, q	Constant, d
Linear Function, $q_0 + q_1 k$	Linear Function, $d_0 + d_1 k$
m^{th} degree polynomial, $q_0 + q_1 k + \cdots + q_m k^m$	m^{th} degree polynomial, $d_0 + d_1 k + \cdots + d_m k^m$
exponential function, qa^k	exponential function, da^k

Likewise, an inhomogeneous term that's a polynomial in k, say $p(k)$, must often be treated as the "exponomial function" $p(k)\ 1^k$.

Let's do 8.3.14(a) together:

14. If $S(n) = \sum_{j=1}^{n} g(j), n \geq 1$, then S can be described with the recurrence relation $S(n) = S(n-1) + g(n)$. For each of the following sequences that are defined using a summation, find a closed form expression:

(a) $S(n) = \sum_{j=1}^{n} j,\ n \geq 1$

The recurrence is

..?..

$S(n) – S(n–1) = n$, or if you prefer,

$S(k) – S(k–1) = k$.

The homogeneous form of the recurrence is

..?..

$S^{(h)}(k) – S^{(h)}(k–1) = 0$.

The characteristic equation is

..?..

$a - 1 = 0$.

The characteristic roots are

..?..

1, with multiplicity

..?..

1.

The general solution $S^{(h)}(k)$ of the homogeneous recurrence is

..?..

$\boldsymbol{S^{(h)}(k) = b\ 1^k = b}$, where b is an unknown constant.

(Check that this makes sense vis-à-vis $S^{(h)}(k) - S^{(h)}(k-1) = 0$.)

Note: We can't find b yet. First we need to find $S^{(p)}(k)$.

The inhomogeneous term in $S(k) - S(k-1) = k$ is

..?..

k.

Table 8.3.17 tells us that the form of a particular solution is

..?..

a first degree polynomial $d_0 + d_1\, k$.

BUT:

..?..

This “is really” $(d_0 + d_1\, k)1^k$, with base 1, so Observation 8.3.23 (which should really be called a Caveat rather than an Observation) tells us that we have to multiply this by

..?..

k^p, where p is

..?..

the multiplicity of 1 as a root of the characteristic equation, which is

..?..

1, so we have $\boldsymbol{S^{(p)}(k) = d_0\, k + d_1\, k^2}$ where d_0 and d_1 are constants.

Let's find the coefficients.

$S^{(p)}(k) - S^{(p)}(k-1) = k$

Method I:

$(d_0\, k + d_1\, k^2) - (d_0\, (k-1) + d_1\, (k-1)^2) = k$

$(d_0\, k + d_1\, k^2) - (d_0\, (k-1) + d_1\, (k^2-2k+1)) = k$

$d_0\, (k - (k-1)) + d_1\, (k^2 - (k^2-2k+1)) = k$

$d_0\, (1) + d_1\, (2k-1) = k$

We want $S^{(p)}(k) - S^{(p)}(k-1) = k$ to be true for all k, so we want $d_0\, (1) + d_1\, (2k-1) = k$ to be true for all k.

Moreover, d_0 and d_1 must be *constants*.

We want $(2d_1)\, k + (d_0 - d_1) = (1)\, k + (0)$.

So we need $2d_1 = 1$ and $d_0 - d_1 = 0$.

So we need $d_1 = 1/2$ and $d_0 = 1/2$.

$\boldsymbol{S^{(p)}(k) = (1/2)\, k + (1/2)\, k^2}$.

$S^{(h)}(k) = b$ and $S^{(p)}(k) = (1/2)\, k + (1/2)\, k^2$, so

$$\boldsymbol{S(k) = S^{(p)}(k) + S^{(h)}(k) = (1/2)\, k + (1/2)\, k^2 + b.}$$

How do we find b?

..?..

Use the initial conditions.

Initial condition: $S(1) = 1$.

General formula with k=1:

$$S(1) = (1/2)\, 1 + (1/2)\, 1^2 + b$$
$$= (1/2) + (1/2) + b$$
$$= 1 + b.$$

So $1 = 1+b$, and $b=0$.

So $S(k) = (1/2)\, k + (1/2)\, k^2 = (k + k^2)/2$.

What else is there to do?

..?..

Check the answer!

The recurrence gives $S(2) = 1+2 = 3$, $S(3) = 1+2+3 = 6$.

The formula gives $S(2) = (2+4)/2 = 3$, $S(3) = (3+9)/2 = 6$.

☺

What was the most algebraically intense part of the derivation?

..?..

I thought the hardest part was solving for $S^{(p)}(k)$.

Here's a way I find easier:

Method II:

We want $S^{(p)}(k) = d_0\, k + d_1\, k^2$ to be a solution to

$S^{(p)}(k) - S^{(p)}(k-1) = k$.

Plug in $k = 1$ and $k = 2$:

$k=1$: $S^{(p)}(1) - S^{(p)}(0) = 1 \Rightarrow (d_0 + d_1) - (0) = 1$

$k=2$: $S^{(p)}(2) - S^{(p)}(1) = 2 \Rightarrow (2d_0 + 4d_1) - (d_0 + d_1) = 2$

So we need $d_0 + d_1 = 1$ and $d_0 + 3d_1 = 2$, i.e. $d_0 = d_1 = 1/2$.

(Not that we don't need to use the initial conditions, even though we're plugging into specific values of k. Any two values would do; 1 and 2 just happen to be especially simple.)