Lecture 18 Slides

Discrete Structures I

Predicates

A **predicate** p over a set U is a function from U to $\{True, False\}$.

Another name for a predicate p over a set U is **unary predicate**.

A function from $U \times U$ to $\{True, False\}$ is known as a **binary predicate**.

Example

“n is prime”

This is a unary predicate over the set of positive integers $\mathbb{P}$

“m and n are relatively prime”

This is a binary predicate over $\mathbb{P}$

Predicates

If $p(x, y)$ is a binary predicate over U, its truth set is

$$T_p = \{(x, y): p(x, y) = True\}$$

which is a subset of $U \times U$.

Makes sense, right?

But some mathematicians *define* the binary relation p as this set!

Example

If S is the set $\{1, 2, 3\}$, the binary relation "$<$" can be uniquely specified by the pairs $(1, 2)$, $(1, 3)$, and $(2, 3)$, since these are precisely the pairs (x, y) in $S \times S$ satisfying $x < y$.

We can therefore *define* the relation "$<$" in $S \times S$ to be

$$\{(1, 2), (1, 3), (2, 3)\}$$

Predicates

More generally, a binary relation r on a set S is defined as a subset of $S \times S$.

For $r \subseteq S \times S$, we write "$a\ r\ b$" (with a, b in S) to mean the assertion that $(a, b) \in r$.

Again more generally, if we have a relation r from one set A to another set B (so that $r \subseteq A \times B$), we write "$a\ r\ b$" (with a in A and b in B) to mean the assertion that $(a, b) \in r$.

Example

We say that "a divides evenly into b" if a and b are integers such that $a \neq 0$ and b/a is an integer. So, if $S = \{1, 2, 3, 4\}$, the relation "divides", as a subset of $S \times S$ is…

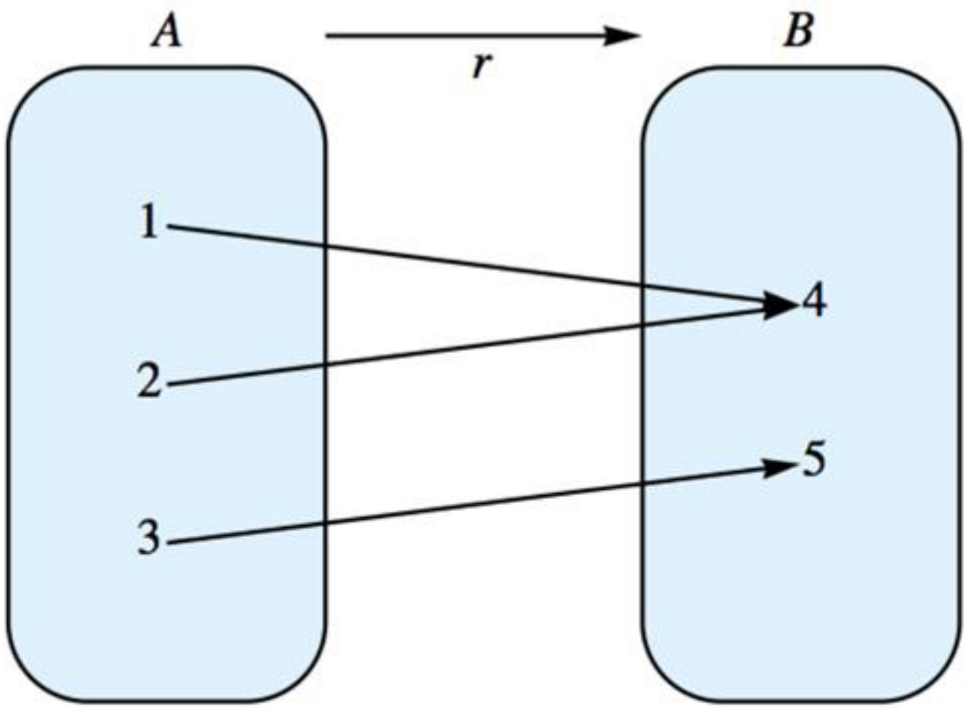
$$\{(1,1), (1,2), (1,3), (1,4), (2,2), (2,4), (3,3), (4,4)\}$$

Digraphs

We can represent a relation from A to B by a directed graph (**digraph**) with arrows from A to B.

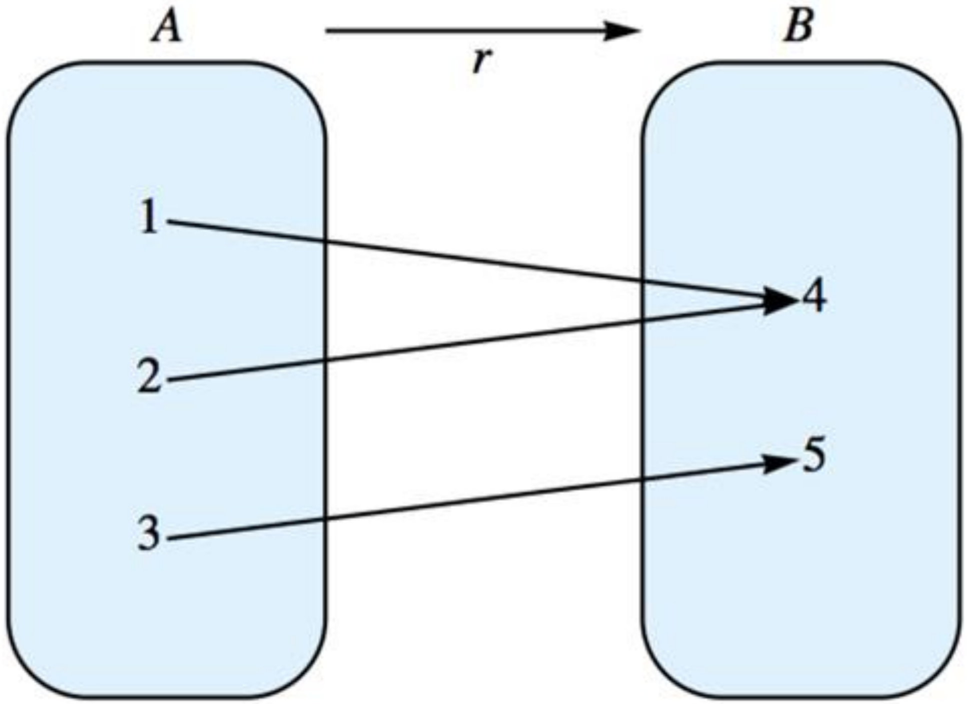
Example

In this image, r, as a subset of $A \times B$, is the set…



Example

In this image, r, as a subset of $A \times B$, is the set $\{(1,4), (2,4), (3,5)\}$.



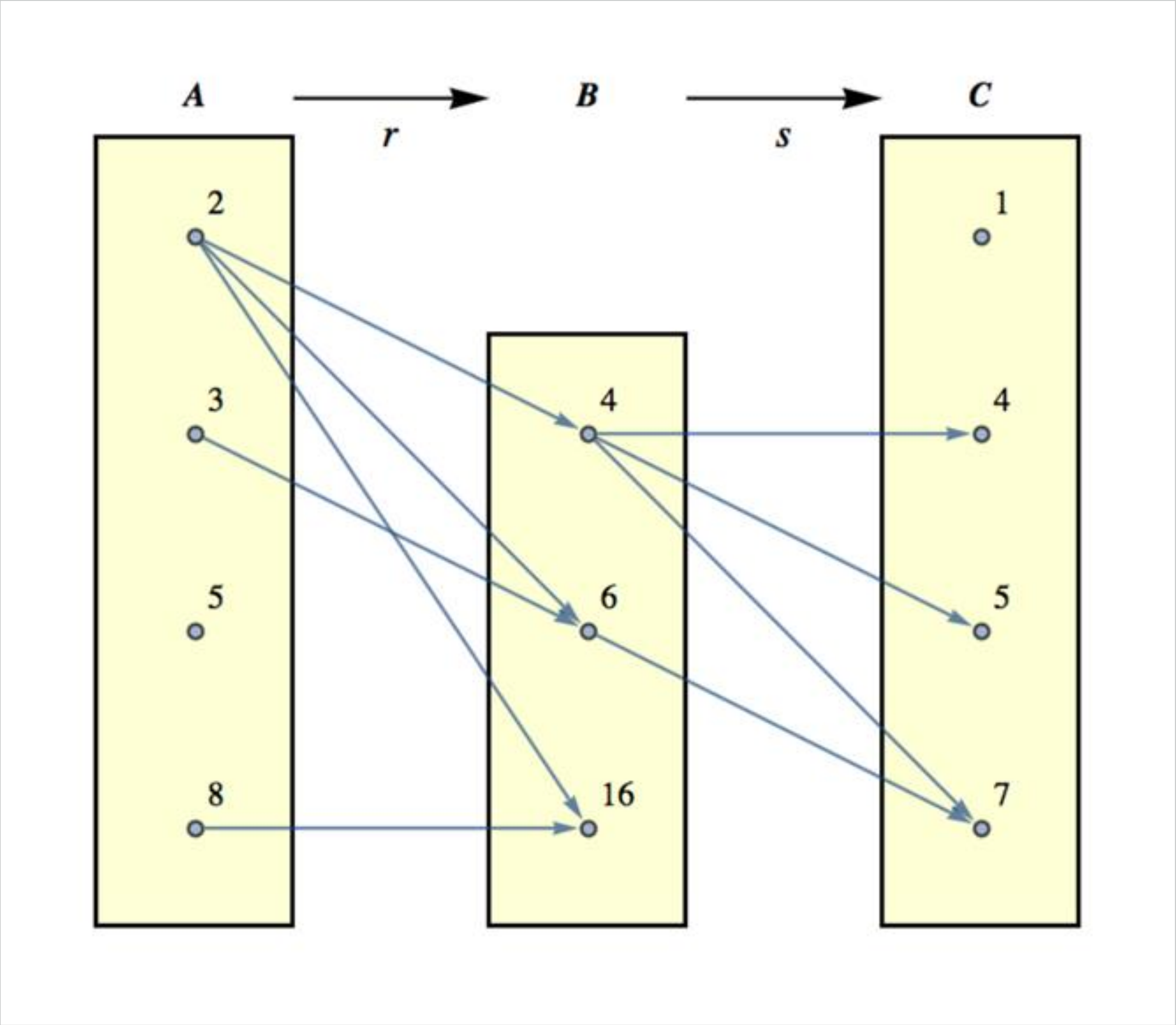
There's a natural way to compose relations:

If r is a relation from A to B and s is a relation from B to C, then rs is a relation from A to C made up of all pairs (a, c) for which there exists some b in B such that $a\ r\ b$ and $b\ s\ c$.

$$r = \{(2,4), (2,6), (2,16), (3,6), (8,16)\}$$

$$s = \{(4,4), (4,5), (4,7), (6,7)\}$$

$$rs = \{(2,4), (2,5), (2,7), (3,7)\}$$



Example

A school in which each child has one or more guardians, each of whom has one or more phone numbers.

$$A = set\ of\ children$$
$$B = set\ of\ parents$$
$$C = set\ of\ parental\ phone\ numbers$$

Note that there may be two paths of length 2 from some child to some phone number if two of the child's parents share a phone number.

http://jamespropp.org/2190/6.1.1.pdf

Example

Let $S = \{1, 2\}$ and r and s be relations from S to S given by $r = \{(1, 1)\}$ and $s = \{(1, 2)\}$.

What is rs?

What is sr?

Are they equal?

$$rs = \{(1, 2)\}$$

$$sr = \{\} = \emptyset$$

Compare

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

but

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

Relations

Note that $\{\}$ is a relation on S; it's just not a very interesting one!

It's the relation on S that is always $False$.

Ditto for $S \times S$ – it is the relation on S that is always $True$.

When we draw a relation on a set (that is, from a set to itself), we can draw to copies of the set as we did above, or we can draw just one.

http://jamespropp.org/2190/6.2.1.pdf

http://jamespropp.org/2190/6.2.3.pdf

Section 6.1

You should be able to:

- Give definitions of basic terms: relation from A into B, relation on set A
- Understand the *divide* relation on $\mathbb{Z}$.
- Visualize a relation using a graph.
- Compute the composition of relations.

Section 6.2

You should be able to:

- Draw digraphs for relations.
- Given the digraph for a relation, write down the set of ordered pairs for the relation.