New operations from old

(Topics: reflexive closure, transitive closure, reflexive-transitive closure, and restriction.)

Suppose r is a relation on A. Recall that this means that r is a subset of $A \times A$; we write "$a\,r\,b$" to mean "$(a, b) \in r$". Note that if r and s are relations on A such that arb implies asb for all $a, b \in A$, then, as subsets of $A \times A$, r is a subset of s. In this situation we often say s **extends** r. For example, if $A = \mathbb{Z}$, the relation $\leq$ extends the relation $<$, because $a < b$ implies $a \leq b$, or equivalently, the set of ordered pairs (a, b) satisfying $a < b$ is a subset of the set of ordered pairs (a, b) satisfying $a \leq b$.

Let i be the relation $\{(a, a) | a \in A\}$ (sometimes called the *identity relation*). If r is any relation on A that is reflexive, it must contain all the ordered pairs (a, a), so it must contain i as a subset. We call i the set-theoretically smallest reflexive relation on A. Then $i \cup r$ is the set-theoretically smallest relation on A that is reflexive and extends r; we call it the **reflexive closure** of r. For example, if $A = \mathbb{Z}$, the reflexive closure of $<$ is $\leq$.

By the definition of composition of relations, $a\,rr\,b$ if and only if $\exists x{\in}A$ such that $a\,r\,x$ and $x\,r\,b$, $a\,rrr\,b$ if and only if $\exists x{,}y{\in}A$ such that $a\,r\,x$ and $x\,r\,y$ and $y\,r\,b$, etc. We write rr as r^2, rrr as r^3, etc. Then $r \cup r^2 \cup r^3 \cup \ldots$ is the set-theoretically smallest relation on A that is transitive and extends r; we call it the **transitive closure** of r and denote it by r^+. Similarly, $i \cup r \cup r^2 \cup r^3 \cup \ldots$ is the set-theoretically smallest relation on A that is reflexive and transitive and extends r; we call it the **reflexive transitive closure** of r and denote it by r^*. For example, if $r = \{(n, n+1) \mid n \in \mathbb{Z}\}$, then the transitive closure of r is the relation $<$ on $\mathbb{Z}$ while the reflexive-transitive closure of r is the relation $\leq$ on $\mathbb{Z}$.

Here is a more concrete way to think about r^+ and r^*. First, $a\,r^+\,b$ if and only if $a\,r^n\,b$ for some $n \geq 1$. That is, $a\,r^+\,b$ if and only if there is a path from a to b consisting of a *positive* number of steps, where each step takes you from one point x to another point y satisfying $x\,r\,y$. Similarly, $a\,r^*\,b$ if and only if there is a path from a to b consisting of a *nonnegative* number of steps.

How do we compute the relation r^+? It helps to define new relations s_n, where $s_n = r \cup r^2 \cup r^3 \cup \ldots \cup r^n$. That is, $a\,s_n\,b$ if and only if there is a path of length 1, 2, 3, ..., or n from a to b, where each step of the path is an r-step.

Claim: $s_{2n} = s_n s_n \cup r$.

Proof (for the curious): If there is a path from a to b of length between

1 and $2n$ (inclusive), then either it is of length 2 or more, in which case we can write the path as a concatenation of two paths, each of length between 1 and n, or else it is of length 1. This shows that s_{2n} is a subset of $s_n s_n \cup r$. To show the reverse direction, note that if we concatenate two paths of length between 1 and n, we get a path of length between 1 and $2n$ (so $s_n s_n$ is a subset of s_{2n}), and that if we take a path of length 1, it's a path of length between 1 and $2n$ (so r is a subset of s_{2n}); it follows that $s_n s_n \cup r$ is a subset of s_{2n}. Since we've shown that s_{2n} and $s_n s_n \cup r$ are subsets of each other, it follows that the two sets are equal, as claimed.

When A is finite, the relations $s_1, s_2, s_4, s_8, \ldots$ eventually become equal to r^+; that's what lies behind the success of Algorithm 6.5.5.

Similar algorithms can be used to compute r^*. One such algorithm treats r^* as the transitive closure of the reflexive closure of r; another treats r^* as the reflexive closure of the transitive closure of r.

If r is a relation on A, so that r is a subset of $A \times A$, and B is a subset of A, then the **restriction** of r to B is defined as $\{(a,b) \in r \ : \ a, b$ are both in $B\}$, or if you prefer, as $\{(a,b) \in B \times B \ : \ a \ r \ b\}$. For example, with $A = \{1,2,3\}$ and $B = \{1,3\}$ the restriction of the relation $\{(1,1),(1,2),(1,3),(2,2),(2,3),(3,3)\}$ to B is the relation $\{(1,1),(1,3),(3,3)\}$.