Divisibility

Why is the divisibility relation a partial ordering on the positive integers (and on any set D_n)?

Recall that $a|b$ iff there exists an integer x such that $ax = b$. Let's restrict to the case where a and b are positive integers, so that the integer x (when it exists) must be positive too.

To show that $|$ is reflexive: We want to show that for all a, a divides a. But $ax = a$ always has the solution $x = 1$.

To show that $|$ is antisymmetric: We want to show that if $a|b$ and $b|a$, then $a = b$.

Suppose $a|b$ and $b|a$. Then $ax = b$ and $by = a$ for positive integers x and y. So $a = by = axy$, so $xy = 1$, implying $x = y = 1$; this implies $a = b$.

To show that $|$ is transitive: We want to show that if $a|b$ and $b|c$ then $a|c$.

Suppose $a|b$ and $b|c$. Then $ax = b$ and $by = c$ for positive integers x and y. We have $a(xy) = (ax)y = by = c$, so xy is a positive integer satisfying $a(xy) = c$, so a divides c.