Zeroeth powers and zero factorial

To see what led mathematicians to define $0!$ as 1, let's consider the similar proposition that $x^0 = 1$ whenever x is a nonzero real number. You might at first have thought that raising a number to the zeroeth power should give zero, and that might be okay if assigning a value to x^0 were an isolated issue, but it's not: we want to define x^n for all integers n (positive, negative, or zero) in a way that makes them fit together well. One way the powers of x with positive exponent already fit together is via the rule $x^{m+n} = x^m \times x^n$; we'll break that rule unless we set x^0 equal to 1, x^{-1} equal to $1/x$, x^{-2} equal to $1/x^2$, and so on. Putting it differently, we could take the proposition that x^{n+1} is equal to x^n times x and turn it around, getting the proposition that x^n is equal to x^{n+1} divided by x. If we want this to continue to hold even when n isn't positive, x^0 "should" be defined as x^1 divided by x, or 1; x^{-1} "should" be defined x^0 divided by x, or $1/x$; x^{-2} "should" be defined to equal x^{-1} divided by x, or $1/x^2$; and so on.

It makes no literal sense to say that $1/100$ is "what you get when you multiply -2 10's together"; when we define 10^{-2} we have to let go of the interpretation of x^n as "what you get when you multiply n x's together" and adopt a more abstract definition of exponentiation. What do we get in return? The very handy modern system of scientific notion for numbers, for starters!

A similar letting-go is required when we try to figure out what $0!$ "should" mean. Instead of thinking of values of $n!$ in isolation, we have to think of how they relate to one another. One way the existing factorial numbers $1!$, $2!$, etc. already fit together is via the rule that $(n+1)!$ is equal to $n!$ times $n+1$. Turning that around, we get the proposition that $n!$ is equal to $(n+1)!$ divided by $n+1$ whenever n is a positive integer. Plugging in $n = 0$ (not a positive integer, but close) we find that $0!$ "should" be $1!$ divided by 1, which is 1. This suggests that we let go of the definition "$n!$ is the product of all the integers from 1 up to n" and make a special rule that $0!$ equals 1.

What do we get from this special proviso? For one thing, the binomial theorem! This theorem (discussed in section 2.4) expands $(x+y)^n$ as a sum of $n+1$ terms of the form $\frac{n!}{k!(n-k)!}x^k y^{n-k}$ (with k running from 0 to n). For instance, $(x+y)^2 = \frac{2!}{2!0!}x^2y^0 + \frac{2!}{1!1!}x^1y^1 + \frac{2!}{0!2!}$. This simplifies to the familiar $x^2 + 2xy + y^2$, but only if $0!$ is taken to be 1. In particular, if we were to take $0!$ to equal 0, then $(x+y)^2$ would expand as $\frac{2}{0}x^2 + \frac{2}{1}xy + \frac{2}{0}y^2$, which makes no sense (since $2/0$ is undefined).