Implication

Logical implication (sometimes called "material implication") is counterintuitive. Regardless of the nature of the propositions p and q, we say that the compound proposition $p \rightarrow q$ is true when p and q are both true, or when p and q are both false, or when p is false and q is true; the only case in which we deem $p \rightarrow q$ to be false is when p is true and q is false.

A compact (but somewhat confusing) way to express $p \rightarrow q$ is $(\neg p) \vee q$. That is, the most efficient way to implement $\rightarrow$ as a circuit uses one NOT-gate and one OR-gate.

A different way to represent $p \rightarrow q$ is to use "disjunctive normal form", and to write $p \rightarrow q$ as a disjunction of conjunctions, namely,

$$(p \wedge q) \vee (\neg p \wedge \neg q) \vee (\neg p \wedge q).$$

(Recall that $\neg$ takes precedence over $\wedge$ and $\vee$.) Notice that this is the definition of $p \rightarrow q$ given in the first paragraph, rendered symbolically.

What about $p \leftarrow q$? $p \leftarrow q$ is equivalent to $q \rightarrow p$, which is equivalent to $\neg q \vee p$, which is equivalent to $p \vee \neg q$.

Alternatively, we can represent $p \leftarrow q$ in disjunctive normal form as $(p \wedge q) \vee (\neg p \wedge \neg q)$, using two NOT gates, two AND gates, and one OR gate.

Finally, what about $p \leftrightarrow q$? One way to write it is as $(p \rightarrow q) \wedge (p \leftarrow q)$. Since $p \rightarrow q$ can be expressed as $\neg p \vee q$, and $p \leftarrow q$ can be expressed as $p \vee \neg q$, we can rewrite $p \leftrightarrow q$ as the conjunction $(\neg p \vee q) \wedge (p \vee \neg q)$, which in terms of circuitry uses two NOT gates, two OR gates, and one AND gate. Notice that this is the conjunctive normal form of $p \leftrightarrow q$, expressing it as a conjunction of disjunctions.

The implications $0 \rightarrow 0$ and $0 \rightarrow 1$ are called "vacuous", and newcomers to mathematical logic are usually puzzled by the cavalier way such statements are accorded the value 1 (that is, true). Why do such a thing?

One reason has to do with the way mathematicians work with the assertion $A \subseteq B$ where A and T are subsets of some universe U. It's often convenient to paraphrase this as "For each $x \in U$, if $x \in A$ then $x \in B$," but this seemingly harmless paraphrase requires us to accept vacuous implications as being true. Consider for instance the case where U is the set of natural numbers, A is the set of multiples of 4 and B is the set of multiples of 2 (that is, the set of even natural numbers).

Now take the proposition "if $x \in A$ then $x \in B$" and replace x by 6; in this case $x \in A$ ("6 is a multiple of 4") is false and $x \in B$ ("6 is a multiple of 2") is true, so we're in the case $0 \to 1$. We'd better accept it as true!

If instead we replace x by 7, then $x \in A$ ("7 is a multiple of 4") is false and $x \in B$ ("7 is a multiple of 2") is false, so we're in the case $0 \to 0$. We'd better accept it as true!

To summarize, if want "For each $x \in U$, if $x \in A$ then $x \in B$" to be true, we have to accept $0 \to 1$ and $0 \to 0$ as having the truth value 1.

Some results of modern mathematics have given new and weirder reasons to accept vacuous assertions. For instance, let p be the assertion that "Generalized Riemann Hypothesis" is true (don't worry about what the GRH is; that's not important here) and let q be the assertion that the "Hecke, Deuring, Mordell, Heilbronn Theorem" is true (don't worry about what that one is either). We don't actually know whether the GRH is true or not, but we know that the HDMHT is true, and the way we know it is very strange: some mathematicians proved that the HDMHT is true if the GRH is *true*, and some other mathematicians proved that the HDMHT is true if the GRH is *false*. In symbols, $p \to q$ and $\neg p \to q$. So, using the logical theorem $((p \to q) \wedge (\neg p \to q)) \to q$, mathematicians concluded that q (that is, the HDMHT) is true.

But isn't this a strange way to know something? Either the GRH is true or the GRH is false, so one of the two propositions $p \to q$ and $\neg p \to q$ must be *vacuous* – we just don't know which! And we need to use both of them to prove the HDMHT! So vacuously true propositions are here to stay, in part because we don't always know ahead of time which propositions are vacuous.

A common logical error is confusing the proposition $p \to q$ with the proposition $q \to p$ (called the *converse* of $p \to q$) or confusing $p \to q$ with $\neg p \to \neg q$ (called the *inverse* of $p \to q$). Completing the quartet of propositions is the converse of the inverse, which asserts $\neg q \to \neg p$; this proposition, which is called the *contrapositive*, can also be thought of as the inverse of the converse of $p \to q$. Every proposition is logically equivalent to its contrapositive; that is, $p \to q$ is true precisely if $\neg q \to \neg p$ is true. Note also that the converse and inverse of $p \to q$ are contrapositives of one another. The relationships between the four propositions are depicted in the diagram below:

$$
\begin{tikzcd}
\boxed{P \rightarrow Q} \arrow[r, dash, "\text{converse}"] \arrow[d, dash, "\text{inverse}"] \arrow[dr, dash, "\text{contrapositive}"] & \boxed{Q \rightarrow P} \arrow[d, dash, "\text{inverse}"] \arrow[dl, dash, "\text{contrapositive}"] \\
\boxed{\neg P \rightarrow \neg Q} \arrow[r, dash, "\text{converse}"] & \boxed{\neg Q \rightarrow \neg P}
\end{tikzcd}
$$