Induction and indexing

The principle of mathematical induction says: If $p(\cdot)$ is a proposition over the universe of positive integers such that $p(1)$ is true and such that $p(n) \Rightarrow p(n+1)$ is true for all $n \geq 1$, then $p(n)$ is true for all n.

Let's expand this: If $p(\cdot)$ is a proposition over the universe of positive integers such that $p(1)$, $p(1) \Rightarrow p(2)$, $p(2) \Rightarrow p(3)$, $p(3) \Rightarrow p(4)$, $\ldots$ are all true, then $p(n)$ is true for all n.

We can re-index this as follows: If $p(\cdot)$ is a proposition over the universe of positive integers such that $p(1)$ is true and such that $p(n-1) \Rightarrow p(n)$ is true for all $n \geq 2$, then $p(n)$ is true for all n.

Sometimes this can be a handier form to use, in terms of keeping the algebra simple.

Let's use the re-indexed form of induction to prove that for all positive integers n, $n(n+1)$ is even.

Claim: For all $n \geq 1$, if $n(n+1)$ is even.

Proof: By induction.

Base case: For $n = 1$, $n(n+1)$ is 2, which is even.

Induction step: Suppose the claim is true for $n-1$ (with $n > 1$). Then $(n-1)n = n^2 - n$ is even. But we know that $2n$ is even, so writing $n(n+1) = n^2+n = (n^2-n)+2n$ we see that $n(n+1)$, being the sum of two even numbers, is even. So the claim is true for n as well.

Since the claim is true for $n = 1$, and since we have shown that whenever the claim is true for $n-1$ the claim is true for n, the claim follows for all n by mathematical induction.

Now let's use the re-indexed form of induction to prove that if S has n elements, then the power set of S has 2^n elements.

Claim: For all $n \geq 0$, if S has n elements then the power set of S has 2^n elements.

Proof: By induction.

Base case: For $n = 0$, the set S must be empty, and the empty set has 1 subset, namely itself. Since $2^0 = 1$, the claim is true in this case.

Induction step: Suppose the claim is true for $n-1$ (with $n > 0$). Consider a set S with n elements. Since $n > 0$, S has at least one element; let x be such an element, and let $S' = S - \{x\}$. Since S' has $n-1$ elements, the claim applies to S', so that S' has 2^{n-1} subsets. Now we will use the law of products (along with our induction hypothesis) to count the subsets of S. Choosing a subset of S amounts to choosing a subset of S' and then deciding whether or not to throw in x. Since choosing a subset of S' can be done in 2^{n-1} ways and choosing whether or not to throw in x can be done in 2 ways, the number of subsets of S (by the law of products) is 2^{n-1} times 2, which equals 2^n. So S has 2^n subsets, as claimed.

Since the claim is true for $n = 0$, and since we have shown that whenever the claim is true for $n - 1$ the claim is true for n, the claim follows for all n by mathematical induction.

Lastly let's use the re-indexed form of induction to prove that

$$1 + 2 + 3 + \ldots + n = n(n + 1)/2$$

for all $n \geq 1$.

Claim: For all $n \geq 1$, $1 + 2 + 3 + \ldots + n = n(n + 1)/2$.

Proof: By induction. To keep the distinction between numbers and propositions straight, let's define $L(n)$ as $1 + 2 + 3 + \ldots + n$ (the left side of the equation we're trying to prove for all n), let's define $R(n)$ as $n(n+1)/2$ (the right side of the equation we're trying to prove for all n), and P_n as the proposition that $L(n) = R(n)$ (the equation we're trying to prove for all n).

Base case: We have $L(1) = 1$ and $R(1) = (1)(1 + 1)/2 = 1$, so $L(1) = R(1)$; therefore the proposition P_1 is true.

Induction step: Suppose P_{n-1} is true, so that $L(n-1) = R(n-1)$. Then $L(n) = 1+2+\ldots+(n-1)+n = (1+2+\ldots+(n-1))+n = L(n-1)+n = R(n-1)+n$ by the induction hypothesis. But $R(n-1) = (n-1)(n-1+1)/2 = (n-1)n/2 = n^2/2-n/2$, so $R(n-1)+n = (n^2/2-n/2)+n = n^2/2+n/2 = (n^2 + n)/2 = n(n + 1)/2 = R(n)$. So we have proved that $L(n) = R(n)$, which proves P_n.

Since P_n is true for $n = 1$, and since we have shown that P_{n-1} implies P_n for all $n > 1$, the claim follows for all $n \geq 1$ by mathematical induction.