Induction and indexing

The principle of mathematical induction says: If $p(\cdot)$ is a proposition over the universe of positive integers such that $p(1)$ is true and such that $p(n) \Rightarrow p(n+1)$ is true for all $n \geq 1$, then $p(n)$ is true for all n.

Let's expand this: If $p(\cdot)$ is a proposition over the universe of positive integers such that $p(1)$, $p(1) \Rightarrow p(2)$, $p(2) \Rightarrow p(3)$, $p(3) \Rightarrow p(4)$, $\ldots$ are all true, then $p(n)$ is true for all n.

We can re-index this as follows: If $p(\cdot)$ is a proposition over the universe of positive integers such that $p(1)$ is true and such that $p(n-1) \Rightarrow p(n)$ is true for all $n \geq 2$, then $p(n)$ is true for all n.

Sometimes this can be a handier form to use, in terms of keeping the algebra simple.

Let's use this form to prove that if S has n elements, then the power set of S has 2^n elements.

Claim: For all $n \geq 0$, if S has n elements then the power set of S has 2^n elements.

Proof: By induction.

Base case: For $n = 0$, the set S must be empty, and the empty set has 1 subset, namely itself. Since $2^0 = 1$, the claim is true in this case.

Induction step: Suppose the claim is true for $n-1$ (with $n > 0$). Consider a set S with n elements. Since $n > 0$, S has at least one element; let x be such an element, and let $S' = S - \{x\}$. There are two types of subsets of S, those that contain x and those that don't. Subsets of S of the second type are just the subsets of S', and since $|S'| = n - 1$, we know there are 2^{n-1} subsets of S of the second type. Subsets of S of the first type are just subsets of S' with x added to them, and they are in one-to-one correspondence with the subsets of S', so there are 2^{n-1} subsets of S of the second type. Adding together the two types of subsets of S, we find that the total number of subsets of S is $2^{n-1} + 2^{n-1} = 2^n$, as claimed.

Since the claim is true for $n = 0$, and since we have shown that whenever the claim is true for $n-1$ the claim is true for n, the claim follows for all n by mathematical induction.