Multiplying matrices

The definition of matrix multiplication that we use turns out to be the right one for a lot of reasons.

One reason is that it lets us solve simultaneous linear equations in several unknowns with a procedure that looks a lot like the one-variable procedure. To solve $ax = b$, we multiply both sides by $a^{-1}$ to get $x = a^{-1}b$ (also known as $b/a$).

To solve

$$
\begin{align*}
ax + by &= e \\
cx + dy &= f
\end{align*}
$$

we write it as

$$
\begin{pmatrix}
a & b \\
c & d
\end{pmatrix}
\begin{pmatrix}
x \\
y
\end{pmatrix} =
\begin{pmatrix}
e \\
f
\end{pmatrix}
$$

or more compactly as $Ax = b$, where $x$ is the vector

$$
\begin{pmatrix}
x \\
y
\end{pmatrix}
$$

and $b$ is the vector

$$
\begin{pmatrix}
e \\
f
\end{pmatrix}
$$

and $A$ is the matrix

$$
\begin{pmatrix}
a & b \\
c & d
\end{pmatrix}.
$$

The way we solve the linear system $Ax = b$ is by “left-multiplying” the LHS and the RHS by the inverse matrix $A^{-1}$ (which we are assuming exists); then

$$
Ax = b
$$

implies

$$
A^{-1}Ax = A^{-1}b
$$

and the left hand side simplifies to $Ix$, which is just $x$, so we get

$$
x = A^{-1}b,
$$
analogous to the formula \( x = a^{-1}b \) we got in the single-variable case. (Note however that we do not write \( A^{-1} \) as \( 1/A \) and we do not write \( A^{-1}b \) as \( b/A \).)

The determinant of
\[
\begin{pmatrix}
a & b \\
c & d
\end{pmatrix}
\]
is \( ad - bc \), and it has the property that for any two 2-by-2 matrices \( A, B \),
\[
det(AB) = det(A) det(B).
\]
In fact, this formula is valid for \( n \)-by-\( n \) matrices, for all \( n \). In the case where \( A \) is invertible and we set \( B = A^{-1} \), we get
\[
det(AA^{-1}) = det(A) det(A^{-1}).
\]
The left hand side simplifies to \( det(I) \), which is 1. So
\[
1 = det(A) det(A^{-1}),
\]
which implies
\[
det(A^{-1}) = 1/ det(A).
\]
In particular, if \( A \) is invertible, its determinant must be non-zero. In fact:

**Theorem:** The square matrix \( A \) has an inverse if and only if \( det(A) \) is not equal to 0.

**Theorem:** If \( A \) and \( B \) are invertible square matrices of the same order, then
\[
(AB)(B^{-1}A^{-1}) = I = (B^{-1}A^{-1})(AB),
\]
implying that \( AB \) is invertible and \( (AB)^{-1} = B^{-1}A^{-1} \).

**Proof:** Use associativity.

\[
(AB)(B^{-1}A^{-1}) = A(BB^{-1})A^{-1} = AIA^{-1} = AA^{-1} = I
\]

and

\[
(B^{-1}A^{-1})(AB) = B^{-1}(A^{-1}A)B = B^{-1}IB = B^{-1}B = I.
\]