

## Multiplying matrices

The definition of matrix multiplication that we use turns out to be the right one for a lot of reasons.

One reason is that it lets us solve simultaneous linear equations in several unknowns with a procedure that looks a lot like the one-variable procedure.

To solve  $ax = b$ , we multiply both sides by  $a^{-1}$  to get  $x = a^{-1}b$  (also known as  $b/a$ ).

To solve

$$\begin{aligned}ax + by &= e \\cx + dy &= f\end{aligned}$$

we write it as

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} e \\ f \end{pmatrix}$$

or more compactly as  $A\mathbf{x} = \mathbf{b}$ , where  $\mathbf{x}$  is the vector

$$\begin{pmatrix} x \\ y \end{pmatrix}$$

and  $\mathbf{b}$  is the vector

$$\begin{pmatrix} e \\ f \end{pmatrix}$$

and  $A$  is the matrix

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

The way we solve the linear system  $A\mathbf{x} = \mathbf{b}$  is by “left-multiplying” the LHS and the RHS by the inverse matrix  $A^{-1}$  (which we are assuming exists); then

$$A\mathbf{x} = \mathbf{b}$$

implies

$$A^{-1}A\mathbf{x} = A^{-1}\mathbf{b}$$

and the left hand side simplifies to  $I\mathbf{x}$ , which is just  $\mathbf{x}$ , so we get

$$\mathbf{x} = A^{-1}\mathbf{b},$$

analogous to the formula  $x = a^{-1}b$  we got in the single-variable case. (Note however that we do not write  $A^{-1}$  as  $1/A$  and we do not write  $A^{-1}\mathbf{b}$  as  $\mathbf{b}/A$ .)

The determinant of

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

is  $ad - bc$ , and it has the property that for any two 2-by-2 matrices  $A, B$ ,

$$\det(AB) = \det(A) \det(B).$$

In fact, this formula is valid for  $n$ -by- $n$  matrices, for all  $n$ . In the case where  $A$  is invertible and we set  $B = A^{-1}$ , we get

$$\det(AA^{-1}) = \det(A) \det(A^{-1}).$$

The left hand side simplifies to  $\det(I)$ , which is 1. So

$$1 = \det(A) \det(A^{-1}),$$

which implies

$$\det(A^{-1}) = 1/\det(A).$$

In particular, if  $A$  is invertible, its determinant must be non-zero. In fact:

**Theorem:** The square matrix  $A$  has an inverse if and only if  $\det(A)$  is not equal to 0.

**Theorem:** If  $A$  and  $B$  are invertible square matrices of the same order, then

$$(AB)(B^{-1}A^{-1}) = I = (B^{-1}A^{-1})(AB),$$

implying that  $AB$  is invertible and  $(AB)^{-1} = B^{-1}A^{-1}$ .

**Proof:** Use associativity.

$$\begin{aligned} (AB)(B^{-1}A^{-1}) &= A(BB^{-1})A^{-1} \\ &= AIA^{-1} \\ &= AA^{-1} \\ &= I \end{aligned}$$

and

$$\begin{aligned} (B^{-1}A^{-1})(AB) &= B^{-1}(A^{-1}A)B \\ &= B^{-1}IB \\ &= B^{-1}B \\ &= I. \end{aligned}$$