Multiplying matrices

The definition of matrix multiplication that we use turns out to be the right one for a lot of reasons.

One reason is that it lets us solve simultaneous linear equations in several unknowns with a procedure that looks a lot like the one-variable procedure.

To solve $ax = b$, we multiply both sides by a^{-1} to get $x = a^{-1}b$ (also known as b/a).

To solve

$$\begin{aligned} ax + by &= e \\ cx + dy &= f \end{aligned}$$

we write it as

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} e \\ f \end{pmatrix}$$

or more compactly as $A\mathbf{x} = \mathbf{b}$, where $\mathbf{x}$ is the vector

$$\begin{pmatrix} x \\ y \end{pmatrix}$$

and $\mathbf{b}$ is the vector

$$\begin{pmatrix} e \\ f \end{pmatrix}$$

and A is the matrix

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

The way we solve the linear system $A\mathbf{x} = \mathbf{b}$ is by "left-multiplying" the LHS and the RHS by the inverse matrix A^{-1} (which we are assuming exists); then

$$A\mathbf{x} = \mathbf{b}$$

implies

$$A^{-1}A\mathbf{x} = A^{-1}\mathbf{b}$$

and the left hand side simplifies to $I\mathbf{x}$, which is just $\mathbf{x}$, so we get

$$\mathbf{x} = A^{-1}\mathbf{b},$$

analogous to the formula $x = a^{-1}b$ we got in the single-variable case. (Note however that we do not write A^{-1} as $1/A$ and we do not write $A^{-1}\mathbf{b}$ as $\mathbf{b}/A$.)

The determinant of

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

is $ad - bc$, and it has the property that for any two 2-by-2 matrices A, B,

$$\det(AB) = \det(A)\det(B).$$

In fact, this formula is valid for n-by-n matrices, for all n. In the case where A is invertible and we set $B = A^{-1}$, we get

$$\det(AA^{-1}) = \det(A)\det(A^{-1}).$$

The left hand side simplifies to $\det(I)$, which is 1. So

$$1 = \det(A)\det(A^{-1}),$$

which implies

$$\det(A^{-1}) = 1/\det(A).$$

In particular, if A is invertible, its determinant must be non-zero. In fact:

Theorem: The square matrix A has an inverse if and only if $\det(A)$ is not equal to 0.

Theorem: If A and B are invertible square matrices of the same order, then

$$(AB)(B^{-1}A^{-1}) = I = (B^{-1}A^{-1})(AB),$$

implying that AB is invertible and $(AB)^{-1} = B^{-1}A^{-1}$.

Proof: Use associativity.

$$\begin{aligned} (AB)(B^{-1}A^{-1}) &= A(BB^{-1})A^{-1} \\ &= AIA^{-1} \\ &= AA^{-1} \\ &= I \end{aligned}$$

and

$$\begin{aligned} (B^{-1}A^{-1})(AB) &= B^{-1}(A^{-1}A)B \\ &= B^{-1}IB \\ &= B^{-1}B \\ &= I. \end{aligned}$$