Posets

Doerr and Levasseur say the relation r on S is antisymmetric if for all a, b in S, $(a\,r\,b$ and $a \neq b) \Rightarrow \neg(b\,r\,a)$.

Because the proposition $(p$ and $\neg r) \Rightarrow (\neg q)$ is equivalent to the proposition $(p$ and $q) \Rightarrow r$ (as we learned from homework #3, problem F), a different (but logically equivalent) way to state the definition is that the relation r on S is antisymmetric if for all a, b in S, $(a\,r\,b$ and $b\,r\,a) \Rightarrow a = b$. I prefer this way of defining "antisymmetric" because it is, well, more symmetric.

A generic partial ordering is often written as $\preceq$ (pronounced "is dominated by"); this is supposed to be reminiscent of the symbols $\leq$ and $\subseteq$. Sometimes it is written as $\leq$, with the understanding that it's not the less-than-or-equal-to relation you know from high school.

For $n \geq 1$, let D_n be the set of positive integer divisors of n; as we saw in div.pdf, the relation "|" (divides) is a partial ordering on this set.

In the divisibility poset D_6, the relation "|", viewed as a set of ordered pairs, consists of (1,1), (1,2), (1,3), (1,6), (2,2), (2,6), (3,3), (3,6), and (6,6).

Definition: If x and y are elements of a poset $[L, \preceq]$, say that $x \prec y$ if $x \preceq y$ and $x \neq y$; say that $x \succeq y$ if $y \preceq x$; and say that $x \succ y$ if $y \prec x$ (as defined above).

Definition: If x and z are elements of a poset L, we say z *covers* x iff $x \prec z$ and there does not exist any y such that $x \prec y$ and $y \prec z$.

Example: In D_6, 6 covers 2 and 3 but not 1.

Theorem: In a finite poset L, $x \preceq y$ iff there exists a sequence $w_1, w_2, \ldots, w_k$ of elements of L with $w_1 = x$ and $w_k = y$, where w_1 is covered by w_2, which is covered by w_3, …, which is covered by w_k. ($k = 1$ in the case $x = y$.)

So we can reconstruct the partial order on L if we know which elements of L cover which other elements of L.

In a Hasse diagram, we draw an edge joining x and y iff y covers x, and we draw y above x.

To recover the Hasse digram from the partial ordering, you may want to use a pencil with an eraser. First draw an arrow from x to y whenever $x \prec y$; then for each existing arrow (say from x to y) delete the arrow from x to y if the other arrows present give use a way to get from x to y in two or more steps. Keep doing this until you get a directed graph that is "anti-transitive"; that is, whenever there is an arrow from x to y and an arrow from y to z, there is *never* an arrow from x to z. This directed graph must be a DAG (directed acyclic graph). Redraw this DAG so that all arrows point upward, and you've got the Hasse diagram.

To recover the partial ordering from the Hasse diagram, we have the following rule: given $a, b \in L$, $a \preceq b$ if and only if there is an upward path in the Hasse diagram from a to b. Note that this includes, as a degenerate case, the situation in which $a = b$ and the path has length 0.