Posets

Doerr and Levasseur say the relation $r$ on $S$ is antisymmetric if for all $a, b$ in $S$, $(a r b$ and $a \neq b) \Rightarrow \neg(b r a)$.

Because the proposition $(p$ and $\neg r) \Rightarrow (\neg q)$ is equivalent to the proposition $(p$ and $q) \Rightarrow r$ (as we learned from homework #3, problem F), a different (but logically equivalent) way to state the definition is that the relation $r$ on $S$ is antisymmetric if for all $a, b$ in $S$, $(a r b$ and $b r a) \Rightarrow a = b$.

A generic partial ordering is often written as $\preceq$ (pronounced “is dominated by”); this is supposed to be reminiscent of (yet distinct from) the symbols $\leq$ and $\subseteq$. Sometimes it is written as $\leq$, with the understanding that it’s not the less-than-or-equal-to relation you know from high school.

For $n \geq 1$, let $D_n$ be the set of positive integer divisors of $n$; as we saw in div.pdf, the relation “$|$” (divides) is a partial ordering on this set.

In the divisibility poset $D_6$, the relation “$|$”, viewed as a set of ordered pairs, consists of $(1,1), (1,2), (1,3), (1,6), (2,2), (2,6), (3,3), (3,6), \text{ and } (6,6)$.

Definition: If $x$ and $y$ are elements of a poset $[L, \preceq]$, say that $x \prec y$ if $x \preceq y$ and $x \neq y$; say that $x \succeq y$ if $y \preceq x$; and say that $x \succ y$ if $y \prec x$ (as defined above).

Definition: If $x$ and $z$ are elements of a poset $L$, we say $z$ covers $x$ iff $x \prec z$ and there does not exist any $y$ such that $x \prec y$ and $y \prec z$.

Example: In $D_6$, 6 covers 2 and 3 but not 1.

Theorem: In a finite poset $L$, $x \preceq y$ iff there exists a sequence $w_1, w_2, \ldots, w_k$ of elements of $L$ with $w_1 = x$ and $w_k = y$, where $w_1$ is covered by $w_2$, which is covered by $w_3, \ldots$, which is covered by $w_k$. ($k = 1$ in the case $x = y$.)

So we can reconstruct the partial order on $L$ if we know which elements of $L$ cover which other elements of $L$.

In a Hasse diagram, we draw an edge joining $x$ and $y$ iff $y$ covers $x$, and we draw $y$ above $x$.

To recover the partial ordering from the Hasse diagram, we have the following rule: given $a, b \in L$, $a \preceq b$ if and only if there is an upward path in the Hasse diagram from $a$ to $b$. Note that this includes, as a degenerate case, the situation in which $a = b$ and the path has length 0.

Equivalently: orient each edge in the Hasse diagram from the lower vertex to the higher vertex. This directed graph can be viewed as a set of ordered pairs, which is to say, a relation. If you take the reflexive-transitive closure of this relation, you get the partial ordering $\preceq$. 
You can read off $\text{glb}(a,b)$ and $\text{lub}(a,b)$ from the Hasse diagram of $L$. $\text{glb}(a,b)$ is the highest element of the Hasse diagram that you can get to by traveling downward from $a$ and also by traveling downward from $b$. Likewise $\text{lub}(a,b)$ is the lowest element of the Hasse diagram that you can get to by traveling upward from $a$ and also by traveling upward from $b$. 