

Posets

Doerr and Levasseur say the relation r on S is antisymmetric if for all a, b in S , $(a r b \text{ and } a \neq b) \Rightarrow \neg(b r a)$.

Because the proposition $(p \text{ and } \neg r) \Rightarrow (\neg q)$ is equivalent to the proposition $(p \text{ and } q) \Rightarrow r$ (as we learned from homework #3, problem F), a different (but logically equivalent) way to state the definition is that the relation r on S is antisymmetric if for all a, b in S , $(a r b \text{ and } b r a) \Rightarrow a = b$.

A generic partial ordering is often written as \preceq (pronounced “is dominated by”); this is supposed to be reminiscent of (yet distinct from) the symbols \leq and \subseteq . Sometimes it is written as \leq , with the understanding that it’s not the less-than-or-equal-to relation you know from high school.

For $n \geq 1$, let D_n be the set of positive integer divisors of n ; as we saw in div.pdf, the relation “|” (divides) is a partial ordering on this set.

In the divisibility poset D_6 , the relation “|”, viewed as a set of ordered pairs, consists of $(1,1)$, $(1,2)$, $(1,3)$, $(1,6)$, $(2,2)$, $(2,6)$, $(3,3)$, $(3,6)$, and $(6,6)$.

Definition: If x and y are elements of a poset $[L, \preceq]$, say that $x \prec y$ if $x \preceq y$ and $x \neq y$; say that $x \succeq y$ if $y \preceq x$; and say that $x \succ y$ if $y \prec x$ (as defined above).

Definition: If x and z are elements of a poset L , we say z covers x iff $x \prec z$ and there does not exist any y such that $x \prec y$ and $y \prec z$.

Example: In D_6 , 6 covers 2 and 3 but not 1.

Theorem: In a finite poset L , $x \preceq y$ iff there exists a sequence w_1, w_2, \dots, w_k of elements of L with $w_1 = x$ and $w_k = y$, where w_1 is covered by w_2 , which is covered by w_3 , \dots , which is covered by w_k . ($k = 1$ in the case $x = y$.)

So we can reconstruct the partial order on L if we know which elements of L cover which other elements of L .

In a Hasse diagram, we draw an edge joining x and y iff y covers x , and we draw y above x .

To recover the partial ordering from the Hasse diagram, we have the following rule: given $a, b \in L$, $a \preceq b$ if and only if there is an upward path in the Hasse diagram from a to b . Note that this includes, as a degenerate case, the situation in which $a = b$ and the path has length 0.

Equivalently: orient each edge in the Hasse diagram from the lower vertex to the higher vertex. This directed graph can be viewed as a set of ordered pairs, which is to say, a relation. If you take the reflexive-transitive closure of this relation, you get the partial ordering \preceq .

You can read off $\text{glb}(a,b)$ and $\text{lub}(a,b)$ from the Hasse diagram of L . $\text{glb}(a,b)$ is the highest element of the Hasse diagram that you can get to by traveling downward from a and also by traveling downward from b . Likewise $\text{lub}(a,b)$ is the lowest element of the Hasse diagram that you can get to by traveling upward from a and also by traveling upward from b .