Proofs

Here’s Example 3.5.7 (a direct proof), done in prose:
Claim: \(p \rightarrow r, q \rightarrow s, p \lor q \Rightarrow r \lor s\). (That is: \((p \rightarrow r) \land (q \rightarrow s) \land (p \lor q) \Rightarrow r \lor s\). Direct proof: Suppose \(p \rightarrow r\), \(q \rightarrow s\), and \(p \lor q\) are all true. Since \(p \lor q\) is true, \(p\) is true or \(q\) is true. Case 1: If \(p\) is true, then the truth of \(p\) combined with the truth of \(p \rightarrow r\) implies the truth of \(r\), which implies the truth of \(r \lor s\). Case 2: If \(q\) is true, then the truth of \(q\) combined with the truth of \(q \rightarrow s\) implies the truth of \(s\), which implies the truth of \(r \lor s\). Since in both cases we have deduced the truth of \(r \lor s\), the truth of the claim follows.

Here’s Example 3.5.17 (an indirect proof), done in prose:
Claim: \(a \rightarrow b, \neg (b \lor c) \Rightarrow \neg a\). Indirect proof: Suppose \(a \rightarrow b\) and \(\neg (b \lor c)\). Suppose furthermore (for purposes of contradiction) that \(\neg a\) is false; that is, suppose \(a\) is true. Since \(a\) is true and \(a \rightarrow b\) is true, \(b\) is true. This implies that \(b \lor c\) is true, implying in turn that \(\neg (b \lor c)\) is false. But since we supposed that \(\neg (b \lor c)\) is true, this is a contradiction. The truth of the claim follows.

Here’s Exercise 3.5.2, done in prose:
We have \((q \land \neg q) \Rightarrow p\), regardless of the nature of the propositions \(p\) and \(q\). That’s because the antecedent is always false, and we’ve defined an implication to be (vacuously) true when the antecedent is false. Here’s a somewhat silly proof in the proof-by-contradiction mode: Suppose that \(q \land \neg q\) is true, and assume for purposes of contradiction that \(p\) is false. Do the assumptions \(q\), \(\neg q\), and \(p\) lead to a contradiction? Yes, because the first two assumptions contradict each other! Hence \(p\) is true.

(I called this proof “silly” because the proof never uses anything about the nature of the proposition \(p\). In fact, the same reasoning that proves \((q \land \neg q) \Rightarrow p\) also proves \((q \land \neg q) \Rightarrow \neg p\)!

And here’s a proof in the style of Doerr and Levasseur: Since \((p \lor q)\) can be deduced from the hypothesis \(q\) and since \(\neg q\) is one of the given hypotheses, the disjunctive simplification rule, applied in the form “\((p \lor q) \land \neg q \Rightarrow p\)”, gives us the conclusion \(p\).

In a similar way, we can check that \(q \Rightarrow (p \lor \neg p)\), regardless of the nature of the propositions \(p\) and \(q\). That’s because the consequent is always true, and an implication is true when the consequent is true.

Group work: 3.5.6:
Let $p = \text{“}x$ does well in discrete math$\text{“}$, $q = \text{“}x$ studies hard$\text{“}$, $r = \text{“}x$ skips classes$\text{“}$, and $s = \text{“}x$ does well in courses$\text{“}$. Then our deduction has the form $p \rightarrow q$, $s \rightarrow \neg r$, $q \rightarrow s \Rightarrow (p \rightarrow (\neg r))$. This is a valid deduction. Proof: Suppose $p \rightarrow q$, $s \rightarrow \neg r$, and $q \rightarrow s$ are all true. To prove that $p \rightarrow (\neg r)$ is true, suppose furthermore that $p$ is true. Since $p$ is true and $p \rightarrow q$ is true, $q$ is true. Since $q$ is true and $q \rightarrow s$ is true, $s$ is true. Since $s$ is true and $s \rightarrow \neg r$, $\neg r$ is true. Thus we have shown that $p$ implies $\neg r$, as was to be shown.

Alternatively, here’s a proof by contradiction: Suppose the hypotheses $p \rightarrow q$, $s \rightarrow \neg r$, and $q \rightarrow s$ are all true, yet the conclusion $p \rightarrow (\neg r)$ is false. The only way $p \rightarrow (\neg r)$ can be false is if $p$ is true and $\neg r$ is false. So we assume that $p$ is true and $\neg r$ is false (the latter of which implies that $r$ is true). Taking stock, we may assume that $p \rightarrow q$, $s \rightarrow \neg r$, $q \rightarrow s$, $p$, and $r$ are all true. But now $p$ and $p \rightarrow q$ give us $q$, and $q$ and $q \rightarrow s$ give us $s$, and $s$ and $s \rightarrow \neg r$ give us $\neg r$, and $\neg r$ contradicts $r$. Having reached the desired contradiction, we have shown that $p \rightarrow q$, $s \rightarrow \neg r$, $q \rightarrow s \Rightarrow (p \rightarrow (\neg r))$ is a tautology.