Proofs

Here's Example 3.5.7 (a direct proof), done in prose:

Claim: $p \to r$, $q \to s$, $p \vee q \Rightarrow r \vee s$. (That is: $(p \to r) \wedge (q \to s) \wedge (p \vee q) \Rightarrow r \vee s$. Direct proof: Suppose $p \to r$, $q \to s$, and $p \vee q$ are all true. Since $p \vee q$ is true, p is true or q is true. Case 1: If p is true, then the truth of p combined with the truth of $p \to r$ implies the truth of r, which implies the truth of $r \vee s$. Case 2: If q is true, then the truth of q combined with the truth of $q \to s$ implies the truth of s, which implies the truth of $r \vee s$. Since in both cases we have deduced the truth of $r \vee s$, the truth of the claim follows.

Here's Example 3.5.17 (an indirect proof), done in prose:

Claim: $a \to b$, $\neg(b \vee c) \Rightarrow \neg a$. Indirect proof: Suppose $a \to b$ and $\neg(b \vee c)$. Suppose furthermore (for purposes of contradiction) that $\neg a$ is false; that is, suppose a is true. Since a is true and $a \to b$ is true, b is true. This implies that $b \vee c$ is true, implying in turn that $\neg(b \vee c)$ is false. But since we supposed that $\neg(b \vee c)$ is true, this is a contradiction. The truth of the claim follows.

Here's Exercise 3.5.2, done in prose:

We have $(q \wedge (\neg q)) \Rightarrow p$, regardless of the nature of the propositions p and q. That's because the antecedent is always false, and we've defined an implication to be (vacuously) true when the antecedent is false. Here's a somewhat silly proof in the proof-by-contradiction mode: Suppose that $q \wedge (\neg q)$ is true, and assume for purposes of contradiction that p is false. Do the assumptions q, $\neg q$, and p lead to a contradiction? Yes, because the first two assumptions contradict each other! Hence p is true.

(I called this proof "silly" because the proof never uses anything about the nature of the proposition p. In fact, the same reasoning that proves $(q \wedge (\neg q)) \Rightarrow p$ also proves $(q \wedge (\neg q)) \Rightarrow \neg p$!)

And here's a proof in the style of Doerr and Levasseur: Since $(p \vee q)$ can be deduced from the hypothesis q and since $\neg q$ is one of the given hypotheses, the disjunctive simplification rule, applied in the form "$(p \vee q) \wedge (\neg q) \Rightarrow p$", gives us the conclusion p.

In a similar way, we can check that $q \Rightarrow (p \vee (\neg p))$, regardless of the nature of the propositions p and q. That's because the consequent is always true, and an implication is true when the consequent is true.

Group work: 3.5.6:

Let p = "x does well in discrete math", q = "x studies hard", r = "x skips classes", and s = "x does well in courses". Then our deduction has the form $p \to q,\ s \to \neg r,\ q \to s \Rightarrow (p \to (\neg r))$. This is a valid deduction. Proof: Suppose $p \to q$, $s \to \neg r$, and $q \to s$ are all true. To prove that $p \to (\neg r)$ is true, suppose furthermore that p is true. Since p is true and $p \to q$ is true, q is true. Since q is true and $q \to s$ is true, s is true. Since s is true and $s \to \neg r$, $\neg r$ is true. Thus we have shown that p implies $\neg r$, as was to be shown.

Alternatively, here's a proof by contradiction: Suppose the hypotheses $p \to q$, $s \to \neg r$, and $q \to s$ are all true, yet the conclusion $p \to (\neg r)$ is false. The only way $p \to (\neg r)$ can be false is if p is true and $\neg r$ is false. So we assume that p is true and $\neg r$ is false (the latter of which implies that r is true). Taking stock, we may assume that $p \to q$, $s \to \neg r$, $q \to s$, p, and r are all true. But now p and $p \to q$ give us q, and q and $q \to s$ give us s, and s and $s \to \neg r$ give us $\neg r$, and $\neg r$ contradicts r. Having reached the desired contradiction, we have shown that $p \to q,\ s \to \neg r,\ q \to s \Rightarrow (p \to (\neg r))$ is a tautology.