Some words about words

Here is some information about some math words, including a few whose use in mathematics is different from their use in ordinary language. If later in the semester you trip across any examples of misleading or obscure math-language that the book doesn't explain well, and you wish that this document had discussed them, let me know, and I'll include them in future versions of this document. (You'll be doing future students a favor **and** you'll get course engagement credit!)

1. **Zero versus the empty set**

Zero is a *number*; the empty set is a *set* with zero elements. Both are handy in similar ways. It's handy to have zero as a *number* so that you can ask "How many elements does this set have?" before you know whether the set actually has any. Likewise, it's handy to have the empty set as a *set* so that you can talk about the intersection of two sets before you know whether the two sets actually intersect. But zero and the empty set are not the same thing. In this class, please write zero as "0" (a circle without a slash) and the empty set as "$\varnothing$" (a circle with a slash) so that it's clear which one you mean. Or you could use set-builder notation and write the empty set as $\{\,\}$.

(Fun fact: In some work on what's called "foundations of math", zero gets defined as the empty set. Ditto for a number system called the *surreal numbers*. But we're not doing foundations of math or studying surreal numbers.)

2. **"Positive", "negative"**

A positive number is one that is greater than 0. A negative number is one that is less than 0. 0 is neither positive nor negative. (Fun fact: In France, at least in some contexts, zero is considered "positif". But we're not in France.)

3. **"Natural"**

In this class (and in most computer science contexts), 0 is considered a natural number (though some mathematicians consider 0 "unnatural" and insist that 1 is the first natural number).

4. **"Even", "Odd"**

Sure, dividing an empty set of cookies into two equally large empty sets doesn't really feel like dividing the empty set in two. But if we were to call zero an odd number, then it would be an odd number than remains odd when you add 1 to it. If we want the odd and even numbers to continue to alternate in a well-behaved way as we move ever further to the left on the number line, we'd better call 0 even, -1 odd, -2 even, and so on.

5. **"Iff"**

"Iff" means if-and-only-if. Example: "You are married if you are a husband" is true, but "You are married iff you are a husband" is false (roughly half of all married people are wives!).

6. **"Divides"**

When a and b are integers with b nonzero, "a divides b" means b is divisible by a, i.e., b/a is an integer. (For instance, 2 and 3 divide 6, but 4 doesn't.) Sometimes people say "a evenly divides b", but don't be misled (by the use of the word "evenly") into thinking that this has something to do with a or b or b/a being even.

Note that "a divides b" says that b divided by a (not a divided by b!) is an integer. Confusing, I know. Just be glad mathematicians don't say "a subtracts b" to mean a is greater than or equal to b or a is less than or equal to b.

By the way, even though 0 isn't odd, it's definitely weird. Even though it's smaller than all the numbers 1, 2, 3, ..., it's also divisible by all of them!

7. **"Let"**

In math, "Let A be a set with $|A| = n$" usually means "Let A be an arbitrary set with $|A| = n$." If the next sentence asks you to prove something about A, you can't just take A to be some particular set with n elements and prove that the claim holds in that one case.

By way of comparison, later on I might ask a question like "Let n be an odd number. Show that $n + 1$ must be even." You would NOT get credit if you wrote "Okay, I'll let $n = 3$, which is odd. Then $n + 1 = 4$, which is even." I would want to see an argument like this: "Suppose n is odd. Then n is of the form $2k + 1$. So then $n + 1$ is of the form $2k + 2$, which can be written as $2(k + 1)$, so it is even." (Here I'm assuming that at that point in the course you would know that a number is even iff it is a multiple of 2 and that a number is odd iff it is 1 more than a multiple of 2.)

8. **"Distinct" versus "unique"**

People new to writing math-prose often confuse these two words.

We say a mathematical entity is *unique* when there's nothing else like it. "There exists a unique real number x such that $x^2 = 0$" means there's only one such number.

We say that two mathematical entities are *distinct* when they are unequal to each other. "There exist distinct real numbers x, y such that $x^2 = y^2$" means you can find real numbers $x \neq y$ satisfying $x^2 = y^2$. More generally, we say that n mathematical entities are distinct when no two of them are equal.

Usually we apply the word "unique" to a single thing, and "distinct" to two or more things.

Here are a few (somewhat artificial) examples to highlight the difference between these two words:

"There exist distinct integers x, y such that $x^2 + y^2 = 1$" is true (just take $x = 0$ and $y = 1$, for instance; there are other solutions but just one will do to prove existence);

"There exist unique integers x, y such that $x^2 + y^2 = 1$" is false (there are four solutions $(x, y) = (0, 1)$, $(0, -1)$, $(1, 0)$, and $(-1, 0)$, though finding just two solutions suffices to prove non-uniqueness);

"There exist distinct integers x, y such that $x^2 + y^2 = 0$" is false; and

"There exist unique integers x, y such that $x^2 + y^2 = 0$" is true.

Moral: "distinct" and "unique" mean very different things.

A symbol we sometimes use to assert "There exists a unique x" is $\exists! x$; for instance, we write $(\exists! x)(\exists! y)\ x^2 + y^2 = 0$.

9. **"Prime", "composite", "relatively prime"**

A *prime* number is an integer greater than 1 that has exactly two positive divisors, namely 1 and itself; the primes as 2, 3, 5, 7, etc.

The integers greater than 1 that have more than two positive divisors (i.e., the ones that can be written in the from $a \times b$ where a and b are integers bigger than 1) are called *composite*.

The integer 1 is considered to be neither prime nor composite.

(Fun fact: In ancient Greece, 1 was considered not to be a number at all. Now of course we think of 1 as a number. Once 1 was accepted as a number, it was natural to think it was prime since it can't be factored, and some older definitions of the word "prime" do indeed apply to 1; but as number theory developed, treating 1 as a prime led to problems, and it was agreed that 1 should be considered to be neither prime nor composite. It took a while for the new definition of "prime" to be adopted universally, and as recently as the early 20th century there were tables of prime numbers that began with 1. But it is no longer the early 20th century.)

Two (not necessarily prime) numbers are called "relatively prime" when they have no common factor. So for instance 10 and 21 are relatively prime (to one another) even though neither of them is prime.

10. **"A"/"an", "the"**

When talking about a mathematical object in the singular we use an indefinite article ("a" or "an") as opposed to the definite article ("the") when we are unsure (or don't wish to specify) whether it is unique. For instance, we wouldn't want to ask "What is the prime factor of 10?" because there's more than one. (Asking "What are the prime factors of 10?" is fine. So is asking "What are the prime factors of 9?" is fine too, since we might not know in advance that there's only one of them.) Likewise, we wouldn't want to say "Let S be the smallest nonempty subset of $\{a, b, c\}$" because we have a three-way tie between the three one-element subsets; instead we would say "Let S be a smallest nonempty subset of $\{a, b, c\}$." It sounds weird in ordinary English, but that's correct math-English.

11. **Capital letters versus lower case letters**

Math needs lots of symbols. To increase the number of symbols available to mean different things, we distinguish between A and a, between B and b, and so on. So please don't thoughtlessly switch between capital and lower case. If a homework problem is about something called n don't rename it N, and vice versa.