Comments on the Midterm Exam

Overall: The average score for the exam was a 74.

People who gave faulty or incomplete arguments but who correctly said "This argument isn't quite right" got more points than people who tried to brazen it out with incorrect proofs presented as if they were correct.

Problem 1: The average score for this problem was 14. (This problem turned out to be too difficult for the exam, so I graded it extra-generously. Without the generosity, the average score on this problem would have been below 10 out of 20.)

Lots of you wrote $A = \{a_1, a_2, a_3, ...\}$, but not every set of real numbers is countable.

Many of you appear to believe that every bounded set has a maximum, or that the least upper bound of a set is always an element of the set, or that the terms "supremum" and "maximum" are synonymous. Make sure you understand why each of these beliefs is false.

Many of you tried to apply Lemma 1.3.7 with $\epsilon = \sup A - \sup B$ to get a direct proof of the result. But note that Lemma 1.3.7 requires that ϵ be positive, and it need not be positive here; for instance consider the case $A = B$. (The case $A = B = (0, 1)$ shows furthermore that $\sup A$ and $\sup B$ might belong to neither A nor B.)

A couple of you found a correct way to apply Lemma 1.3.7 to this problem, namely, in the context of a proof by contradiction. Suppose that $\sup A$ is less than $\sup B$. Take $\epsilon = \sup B - \sup A$; by Lemma 1.3.7, there exists $b \in B$ with $b > \sup B - \epsilon = \sup A$. But since $B \subseteq A$, b belongs to A, so $b \leq \sup A$. This is a contradiction. (If all the details were present, this proof got 20 points out of 20.)

Problem 2: The average score for this problem was 15.

For part (a), some of you wrote a proof by induction that said "Suppose $x_n \leq 6$ for all n. Then [details omitted] so $x_{n+1} \leq 6$ for all n. Hence by induction $x_n \leq 6$ for all n." Make sure you understand why this is a bad way to write a proof by induction. (The first and second appearances of the phrase "for all n" is the problem.)

One student derived (a) as a consequence of (b)! That is, the student proved (b) by induction (as I requested) and then observed that $x_n \leq x_{n+1}$ implies $x_n \leq 6$. (Check: $x_n \leq x_{n+1} = (2x_n + 6)/3$ implies $3x_n \leq 2x_n + 6$

which implies $x_n \leq 6$.) I gave this solution full credit, even though what I'd wanted was two separate induction arguments (one for (a) and one for (b)).

For part (c), some of you wrote "Since it's bounded by 6, the limit is 6." This does not follow. After all, the sequence is also bounded by 7, but the limit isn't 7!

The correct way to find the limit L (once you've proved that it exists by citing the Monotone Convergence Theorem) is to apply the algebraic limit theorem to the recursive equation $x_{n+1} = (2x_n+6)/3$ to obtain $L = (2L+6)/3$ and solve. (One student did a variant of this, starting from the equation $6 = (2L+6)/3$. However, this solution did not get full credit, because its logic is circular. That is, there's no reason to assume that the limit L should satisfy $(2L+6)/3 = 6$; the only way to prove this is by assuming that $L = 6$, which is what we're trying to prove.)

Here's an alternate method of proving that the limit is 6 that one student found: Since the sequence is increasing, the limit of the sequence (x_n) equals the supremum of the set of values that x_n takes on. Let $s = \sup\{x_1, x_2, \ldots\}$. Given $\epsilon > 0$, there exists x_n so that $x_n > s-\epsilon$. Then $s \geq x_{n+1} = (2x_n+6)/3 > (2(s-\epsilon)+6)/3$, so $3s > 2(s-\epsilon)+6 = 2s+6-2\epsilon$, so $s > 6-2\epsilon$. Since this is true for all ϵ, $s \geq 6$. But we already know that that 6 is an upper bound for the sequence, so $s \leq 6$. Combining the inequalities, we get $s = 6$.

Problem 3: The average score for this problem was 16.

Some of you wrote things like "For every $\epsilon > 0$ and for some $N \in \mathbf{N}$, when $n \geq N$, [etc.]" This makes it sound like there's an N that works for every ϵ, which is not the case. Better to say "For every $\epsilon > 0$ there is some $N \in \mathbf{N}$ such that when $n \geq N$, [etc.]" which makes it clear that N may depend on ϵ.

A few people wrote "Let $N = 9/\epsilon^2 - 4$" (I'm guessing it was people who skipped the October 21 class since I specifically addressed that mistake during my lecture and urged you all not to make it). It's a mistake because $9/\epsilon^2 - 4$ is usually not an integer. For full credit, you needed to say "Take $N > 9/\epsilon^2 - 4$" (which tacitly appeals to the Archimedean property).

Problem 4: The average score for this problem was 16.

Many of you noticed something that I missed when writing the exam: Lemma 2.6.3, which says that Cauchy sequences are bounded! So that's an even simpler proof than the proof I gave for this direction.

Problem 5: The average score for this problem was 13.

Don't confuse limits (of sequences) and limit points (of sets). E.g., 3 is the limit of the sequence $1, 2, 3, 3, 3, 3, \ldots$ but 3 is not a limit-point of the set $\{1, 2, 3\}$.

Some of you, instead of citing Theorem 3.2.14(ii), proved from scratch that the intersection of two closed sets is closed. Here's how the argument goes: "If K and F are closed, then K contains its limits and F contains its limits, therefore $K \cap F$ contains its limits, so $K \cap F$ is closed." I didn't give this solution full credit, because the "therefore" step requires a more careful proof. Here are more details on the "therefore" step: Suppose x is a limit point of $K \cap F$. That means that every neighborhood of x contains a point in $K \cap F$ other than x itself. Since a point that's in $K \cap F$ is in K, this implies that every neighborhood of x contains a point of K other than x itself. Hence x is a limit point of K. Since K is closed, x is in K. The same argument, with F playing the role of K, shows that x is in F. Hence x is in $K \cap F$, as we needed to show.

Although my original solution separated the problem into several parts (show that $K \cap F$ is closed, show that $K \cap F$ is bounded, and then apply Theorem 3.3.4), some of you took a different approach and worked directly with the definition of compact sets. Here's the argument: Suppose K is compact and F is closed. Consider a sequence (x_n) in $K \cap F$. Since (x_n) is a sequence in K, and K is compact, (x_n) has a subsequence (x_{n_k}) that converges to an element x of K. However, since (x_n) is a sequence in F, the convergent subsequence (x_{n_k}) is also a sequence in F, and since F is closed, the limit $\lim x_{n_k} = x$ must also be in F. Since x is in both K and F, it is in $K \cap F$. Thus we have shown that every sequence in $K \cap F$ has a subsequence that converges to a limit that is also in $K \cap F$. This verifies that $K \cap F$ satisfies the definition of compactness.