Comments on the Final Exam

Overall: The average score for the exam was 82.

People who gave faulty or incomplete arguments but who correctly said "This argument isn't quite right" got more points than people who presented incorrect proofs as if they were correct.

Problem 1: The average score for this problem was 15.8.

I thought Problem 1 would be easy, since it's essentially the same as exercise 4.3.8 from Abbott (which you handed in for assignment 7), only with the irrational numbers taking the role played by the rational numbers in exercise 4.3.8. However, most of you had trouble with it, and even those of you who correctly realized that it had something to do with density frequently wrote that the density of the rational numbers in $\mathbf{R}$ was relevant, whereas it's the density of the *irrational* numbers that's needed here. (Some of you hedged your bets and wrote "Since the rationals and irrationals are both dense in $\mathbf{R}$...".) So I ended up grading this problem more generously than I thought I would have to.

A couple of you cited the result of Exercise 4.3.7. Technically this isn't allowed, since I stated on the front page of the exam "Unless the statement of a problem explicitly says otherwise, your solution to the problem may appeal to any facts that are stated in the text or were discussed in class, aside from homework problems." (You may recall that Exercise 4.3.7 was not one of the exercises I proved in class; it was assigned in homework set 7. So I didn't want you to use it.) However, I now realize that I never really made this distinction clear, or suggested that your cheat-sheet should distinguish between homework and non-homework facts. And I certainly can't expect you to remember which facts were covered in class and which ones were covered in homework! So I decided to be lenient and to accept the solution which says "By Exercise 4.3.7, the set $K = \{x \in \mathbf{R} : f(x) = 0\}$ must be a closed set containing the irrationals, and since the closure of the irrationals is all of $\mathbf{R}$, K must be $\mathbf{R}$."

The students who didn't recall the relevance of density ended up trying to solve this problem by going back to the definition of continuity. However, without the key fact that the irrationals are dense in $\mathbf{R}$, this approach won't work. Some of you brazened it out anyway, applying the inequality $|f(x) - f(c)| < \epsilon$ without first checking that $|x - c| < \delta$; a few of you also seemed to think that δ depends only on ϵ and not on ϵ and c (i.e., you assumed that f is uniformly continuous).

As always, you got more points if you acknowledged these defects in your proofs ("this step seems iffy to me"), and even more points if you pinpointed the defect ("this step assumes uniform continuity; I didn't have time to find a proof with ϵ depending on c as well as δ").

Note that if we make the assumption that f is uniformly continuous, then we can argue as follows: "Fix $\epsilon > 0$, and take $\delta > 0$ so that for all x, y in $\mathbf{R}$, $|x - y| < \delta$ implies $|f(x) - f(y)| < \epsilon$. Given a real number x, the density of the irrationals implies that we can find an irrational number y with $|x - y| < \delta$, so $|f(x)| = |f(x) - f(y)| < \epsilon$. Hence f only takes on values in $(-\epsilon, \epsilon)$. And since ϵ was arbitrary, f only takes on the value 0." (In fact, this can be made into a fully valid proof if one restricts f to compact sets. Specifically, for any $M > 0$, the restriction of f to the compact set $[-M, M]$ is uniformly continuous, so the above argument can be applied to show that f vanishes on $[-M, M]$; and since M was arbitrary, f vanishes everywhere.)

A common mistake was to switch quantifiers, and to argue like this: "Let c be some fixed irrational number. Then for any fixed $\epsilon > 0$, every rational number x sufficiently close to c satisfies $|f(x)| < \epsilon$. Therefore, every rational number sufficiently close to c satisfies $|f(x)| < \epsilon$ for every ϵ, so that $f(x) = 0$." The problem with this argument is that "every rational number x sufficiently close to c" really means "every rational number x in the interval $(c-\delta, c+\delta)$", where δ depends on ϵ; if you want to conclude that $|f(x)|$ is less than EVERY positive ϵ, you need to choose the rational number x to be in the intersection of all the intervals $(c - \delta, c + \delta)$. But the intersection of these intervals is just the singleton set $\{c\}$, which doesn't contain any rational numbers at all!

A couple of you leaned towards a different valid solution, which goes like this: "Suppose $f(x)$ is not zero for all x. Assume we have c with $f(c) > 0$ (the case where $f(c) < 0$ is similar). Let $\epsilon = f(c) > 0$. Since f is continuous at c, there exists $\delta > 0$ such that for all x with $|x-c| < \delta$ we have $|f(x)-f(c)| < \epsilon$, implying $f(x) > 0$. But the density of the irrationals implies that there exists an irrational number x satisfying $|x-c| < \delta$, and the irrationality of x implies that $f(x) = 0$. This contradicts the inequality $f(x) > 0$."

Problem 2: The average score for this problem was 16.

This was also a harder problem for you than I expected. I assumed that you'd notice the resemblance to Extra Problem A from assignment 9, and apply the same sort of reasoning, first applying the Mean Value Theorem to

f (or g) on $[0, 1]$ and $[0, 2]$, and then applying the MVT to f' (or g') on $[c_1, c_2]$, where c_1 and c_2 are the numbers given to you by the first two applications of the MVT. Instead, many of you leaped in by applying the Mean Value Theorem to f (or g) on the interval $[0, 2]$, which is a dead end.

One student got an almost correct solution, but lost a couple of points for an interesting sort of mistake. Note that if you find c_1 and c_2 correctly but only record the fact that c_1 and c_2 belong to $(0, 2)$, then you can't immediately conclude that c_1 and c_2 are unequal; and if you don't know $c_1 \neq c_2$, you can't apply the MVT to the function g on the interval between c_1 and c_2. Fortunately, it's easy to fix this, by noting that the first two applications of the MVT yields numbers c_1, c_2 with c_1 in $(0, 1)$ and c_2 in $(1, 2)$, so that $c_1 < 1 < c_2$.

Problem 3: The average score for this problem was 17.6.

A couple of you confused uniform convergence with uniform continuity, or assumed that all of the functions f_n, f, g_n, g are continuous. There are no assumptions about continuity in the statement of the problem.

Some of you wrote "$|f_n(x) - f(x)| < \epsilon/2$ for all x" where others wrote simply "$|f_n - f| < \epsilon/2$". Either is fine with me, but if you're at all shakey with proofs, it's better to write the former, since it makes the quantification explicit. One student adopted a mixed notation, writing $|f_n(x) - f| < \epsilon/2$, which later led him to express qualms about what was essentially a complete and correct solution.

One student tried to apply Corollary 4.2.4(ii); but, correctly applied, the Corollary only tells us that $f_n + g_n$ converges to $f + g$ *pointwise*, not uniformly. Similarly, another student wrote "By the algebraic limit law, $f_n + g_n$ converges to $f + g$ uniformly," but you will search in vain for an algebraic limit law in chapter 6 analogous to the ones that appear in chapter 2, 4, and 5. In fact, one can easily formulate and prove such algebraic limit laws for uniform continuity, but Abbott didn't do it, so you can't cite such a result from Abbott.

Problem 4: The average score for this problem was 16.7.

Most of you got the correct implications (a) implies (b) implies (d) implies (c), citing Theorems 6.5.2, 6.5.1 and 6.5.2, respectively. However, most of you neglected to prove that (b) does not imply (a), that (d) does not imply (a) or (b), and that (c) does not imply (a), (b), or (d) (by means of suitable counterexamples, as stated in the problem). See my solutions-sheet for appropriate counterexamples.

Note that once we know that (a) implies (b) but (b) does not imply (a), that (b) implies (d) but (d) does not imply (b), and that (d) implies (c) but (c) does not imply (d), all the other implications and non-implications follow by pure logic. E.g., once we know that (a) implies (b) and (b) implies (d), it follows that (a) implies (d). Likewise, to show that (d) does not imply (a), note that if (d) did imply (a), then the fact that (a) implies (b) would make (d) imply (b); but we already know that (d) does not imply (b).

An alternative counterexample for showing that (d) does not imply (a) or (b) is $\sum_{n=0}^{\infty} \frac{1}{(n+1)^2} x^n$.

Problem 5: The average score for this problem was 16.1.

Some of you confused $L(f,P)$ with $L(f)$ and $U(f,P)$ with $U(f)$. To answer this problem, it's not enough to consider $L(f,P)$ and $U(f,P)$ for particular partitions P; you have to take the supremum and infimum of these respective quantities as P varies over all partitions. So if you just computed $L(f,P_n)$ and $U(f,P_n)$ where P_n is the partition of $[0,1]$ into n equal-width subintervals, you didn't get full credit. Some of you took the limit of $L(f,P_n)$ and $U(f,P_n)$ as $n \to \infty$, but how do you know that they converge to $L(f,P)$ and $U(f,P)$ respectively? Abbott doesn't give any theorems that guarantee this (the closest he comes is Theorem 8.1.2, which we didn't cover).

Some of you got the correct answers $L(f) = 0$ and $U(f) = \frac{1}{2}$, but unless you provide rigorous justification for these answers, I can't give full credit.

You'll note that my solution said "Since the rationals are dense in $[0,1]$, $\sup\{f(x) : x_{k-1} \le x \le x_k\} = x_k = \sup\{x : x_{k-1} \le x \le x_k\}$ for every subinterval $[x_{k-1}, x_k]$ of every partition P." In fact, the first equality requires a little more argument than I included in my write-up. It's clear that $f(x) \le x \le x_k$ for all x in $[x_{k-1}, x_k]$, so $\sup\{f(x) : x_{k-1} \le x \le x_k\}$ is at most x_k. On the other hand, for every $\epsilon > 0$, there exists a rational number x in $[x_k - \epsilon, x_k]$, and for this x we have $f(x) = x \ge x_k - \epsilon$. Hence $\sup\{f(x) : x_{k-1} \le x \le x_k\}$ is at least $x_k - \epsilon$. Since this is true for all ϵ, we have $\sup\{f(x) : x_{k-1} \le x \le x_k\} \ge x_k$, and since we already have $\sup\{f(x) : x_{k-1} \le x \le x_k\} \le x_k$, we conclude that $\sup\{f(x) : x_{k-1} \le x \le x_k\} = x_k$, as claimed.

A couple of you gave a variant of the preceding argument, asserting merely that $[x_{k-1}, x_k]$ (rather than $[x_k - \epsilon, x_k]$) contains a rational number; from this we get the weaker estimate $\sup\{f(x) : x_{k-1} \le x \le x_k\} \ge x_{k-1}$. It may be possible to use this estimate to get a different proof of the formula $U(f) = \frac{1}{2}$, but it would require more work than the method I gave in my solution.