Boolean algebras

BIG FACT (Theorem 13.4.6): All finite Boolean algebras are isomorphic to (i.e., "look like") the inclusion lattice of the power set of some finite set S.

We say that the lattice $[L; \vee, \wedge]$ is isomorphic to the lattice $[L'; \vee', \wedge']$ if there is a bijection f from L to L' such that for all x, y in L, $f(x \vee y) = f(x) \vee' f(y)$ and $f(x \wedge y) = f(x) \wedge' f(y)$.

D_{30} (the set of divisors of 30 ordered by divisibility) is a Boolean algebra; 1 is the bottom element (the "**0**"), 30 is the top element (the "**1**"), meet corresponds to gcd, join corresponds to lcm, and the complement of a is $30/a$.

D_{30} is isomorphic to the power set of $\{2, 3, 5\}$ ordered by inclusion. The isomorphism from D_{30} to $\mathcal{P}(2, 3, 5)$ sends an integer n to the set of its prime divisors. The reverse isomorphism from $\mathcal{P}(2, 3, 5)$ to D_{30} sends every subset of $\{2, 3, 5\}$ to the product of the elements of the subset. Note that this requires that we treat the product of the elements of the empty set as 1; for more on this, see http://jamespropp.org/2190/factorial.pdf.

More generally, D_n (ordered by divisibility) is a Boolean algebra whenever n is squarefree (i.e., has no repeated prime factors), and is isomorphic to the inclusion lattice of $\mathcal{P}(\{p, q, \ldots, r\})$, where $\{p, q, \ldots, r\}$ is the set of prime divisors of n.

What about nonsquarefree values of n? When n is a product of powers of two different primes, say $p^i q^j$, then D_n is isomorphic to $(A_i \times A_j, \preceq)$, in the notation of problem 13.1.6. Likewise, when n is $p^i q^j r^k$ with p, q, r distinct primes, then D_n is isomorphic to $A_i \times A_j \times A_k$, equipped with the partial ordering in which $(a, b, c) \preceq (a', b', c') \Leftrightarrow a \leq a'$ and $b \leq b'$ and $c \leq c'$.