Eigenvalues and eigenvectors

Background terminology worth reviewing: When f is a function (which we also call a mapping or a map) from a set A to a set B, with a in A and b in B satisfying $f(a) = b$, we often say that f "sends" a to b, or "carries" a to b, or "maps" a to b.

Diagonal matrices: Consider the map f from $\mathbb{R}^2$ to $\mathbb{R}^2$ that sends (x, y) to $(2x, 2y)$, for all real numbers x, y. It just stretches the whole plane uniformly by a factor of 2. Every vector gets stretched by a factor of 2 and doesn't change direction. Write (x, y) and $(2x, 2y)$ as column vectors:

$$\begin{pmatrix} 2x \\ 2y \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

Now consider the linear map from $\mathbb{R}^2$ to $\mathbb{R}^2$ that sends (x, y) to $(2x, 3y)$, for all real numbers x, y. It stretches horizontal vectors by a factor of 2 (leaving them pointing in the same direction), and stretches vertical vectors by a factor of 3 (leaving them pointing in the same direction). Other vectors don't get stretched uniformly; they get stretched more in the vertical direction than the horizontal direction. The only nonzero vectors that get stretched without changing direction are horizontal vectors and vertical vectors. That is, the "eigendirections" are the horizontal and vertical directions. In terms of matrices, we have

$$\begin{pmatrix} 2x \\ 3y \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

One of the main themes of linear algebra is that for most linear maps, there are eigendirections that are worth finding, but they usually aren't pointing in the obvious coordinate directions. Finding those eigendirections is related to the process of diagonalizing the matrix associated with the linear map.

Diagonalizing a matrix A means finding an invertible matrix P such that $P^{-1}AP$ is a diagonal matrix D. Equivalently, $A = PDP^{-1}$. It's easy to raise a diagonal matrix D to the nth power (where n is a positive integer): just raise each diagonal entry to the nth power. E.g., if

$$D = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$$

then

$$D^n = \begin{pmatrix} a^n & 0 \\ 0 & b^n \end{pmatrix}$$

This gives us a way to raise A to the nth power, by means of the formula $A^n = PD^nP^{-1}$. To see why this formula is true, let's check the case $n = 2$: since $A = PDP^{-1}$,

$$\begin{aligned} A^2 &= (PDP^{-1})(PDP^{-1}) \\ &= PD(P^{-1}P)DP^{-1} = PDIDP^{-1} \\ &= PDDP^{-1} \\ &= PD^2P^{-1}. \end{aligned}$$

Using this, let's check the case $n = 3$:

$$\begin{aligned} A^3 &= A^2\,A \\ &= (PD^2P^{-1})(PDP^{-1}) \\ &= PD^2(P^{-1}P)DP^{-1} \\ &= PD^2IDP^{-1} \\ &= PD^2DP^{-1} \\ &= PD^3P^{-1}. \end{aligned}$$

And we can prove the formula for all larger n by induction.

(Forgotten how to compute P^{-1}? See Chapter 5. In the case $n = 2$, the method of Example 5.2.10 is very handy.)

Eigenvectors: The following are equivalent:

$$\begin{aligned} A\mathbf{v} &= \lambda\mathbf{v} \\ A\mathbf{v} &= \lambda I\mathbf{v} \\ A\mathbf{v} - \lambda I\mathbf{v} &= \mathbf{0} \\ (A - \lambda I)\mathbf{v} &= \mathbf{0} \end{aligned}$$

We say such a vector $\mathbf{v}$ (with $\mathbf{v} \neq \mathbf{0}$) is an eigenvector of the matrix A associated with the eigenvalue λ.

Let A be the 2-by-2 matrix $\begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}$. We see that every vector of the form $\begin{pmatrix} c \\ 0 \end{pmatrix}$ is an eigenvector for the eigenvalue 2, while every vector of the form $\begin{pmatrix} 0 \\ c \end{pmatrix}$ is an eigenvector for the eigenvalue 3.

Is there a non-zero vector whose length gets stretched by $5/2$ (a stretching factor between 2 and 3) when we multiply that vector (on the left) by A? If we set up the equation

$$\begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} (5/2)x \\ (5/2)y \end{pmatrix}$$

as the system of equations

$$\begin{aligned} 2x &= (5/2)x, \\ 3y &= (5/2)y, \end{aligned}$$

we find that the only solution is $x = y = 0$.

In fact, you can replace $5/2$ here by any number λ other than 2 or 3, and you'll find that the equation

$$\begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \lambda x \\ \lambda y \end{pmatrix}$$

has only the trivial solution $x = y = 0$.

This is the typical situation: an n-by-n matrix has at most n eigenvalues, and if you try to solve the eigenvector equation $A\mathbf{v} = \lambda\mathbf{v}$ when λ isn't an eigenvalue, the only solution you'll find is $\mathbf{v} = \mathbf{0}$.

When the matrix $A - \lambda I$ has non-zero determinant, there cannot be any eigenvectors associated with λ. For, if $\det(A - \lambda I)$ is non-zero, $A - \lambda I$ has an inverse matrix B, so that $(A - \lambda I)\mathbf{v} = \mathbf{0}$ implies $B(A - \lambda I)\mathbf{v} = B\mathbf{0}$. The left hand side reduces to $I\mathbf{v}$, which equals $\mathbf{v}$, while the right hand side is $\mathbf{0}$, so we get $\mathbf{v} = \mathbf{0}$. And by definition, eigenvectors are non-zero. So, turning this around, the only way λ can be an eigenvalue of A is if $\det(A - \lambda I) = 0$.

We have shown that IF λ is an eigenvalue of A, THEN $\det(A - \lambda I) = 0$. Furthermore the converse is true: IF $\det(A - \lambda I) = 0$, THEN λ is an eigenvalue of A.

Not all matrices can be diagonalized. For instance,

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

cannot be diagonalized if one restricts to matrices with real-number entries, but it can be diagonalized if one permits complex-number entries; its diagonal form is

$$\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$$

where i is the square root of -1. And the matrix

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

cannot be diagonalized, no matter what sorts of numbers one allows to serve as entries. Let's see why: The equation

$$\begin{pmatrix} 0-\lambda & 1 \\ 0 & 0-\lambda \end{pmatrix}$$

has determinant λ^2, so the only eigenvalue is 0 (a double eigenvalue because 0 is a double root of the eigenvalue equation $\lambda^2 = 0$). Let's find all eigenvectors for the eigenvalue $\lambda = 0$. The equation $A\mathbf{v} = \lambda\mathbf{v}$ becomes

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 0\begin{pmatrix} x \\ y \end{pmatrix}$$

or

$$\begin{pmatrix} y \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

which has the one-parameter family of solutions $x =$ anything, $y = 0$. So we have only a one-dimensional eigenspace for the eigenvalue 0, and that's the only eigenvalue there is for this matrix, so we can't find two linearly independent eigenvectors, which implies that the matrix isn't diagonalizable.

When one diagonalizes a matrix A, one gets to choose in what order to list the eigenvalues; the eigenvalues will appear in that order on the diagonal of D. So for instance when you "diagonalize" the already diagonal matrix

$$\begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}$$

using $\lambda_1 = 3$ and $\lambda_2 = 2$, you'll get the new diagonal form

$$\begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix}.$$

You can check that you'll get this as D regardless of which eigenvector $\mathbf{v}_1$ you take (associated with the eigenvalue λ_1) and which eigenvector $\mathbf{v}_2$ you take (associated with the eigenvalue λ_2).