Homomorphism of groups

Remember that an isomorphism from a group $[G_1, *_1]$ to a group $[G_2, *_2]$ is a bijection f from G_1 satisfying the relation

(*) $f(a *_1 b) = f(a) *_2 f(b)$ for all a, b in G_1.

If we *drop* the requirement that f be bijective, then what we have is the notion of a *homomorphism.* For instance, there is no bijection from $\mathbb{Z}$ to $\mathbb{Z}_2$, so there certainly isn't an isomorphism, but there is a map f from $\mathbb{Z}$ to $\mathbb{Z}_2$ that sends the even integers to 0 and the odd integers to 1, and it has property (*).

You can check that $f(a+b) = a +_2 b$ for all integers a, b. This means that if you want to know the remainder when you divide $a+b$ by 2, compute the remainder r that you get when you divide a by 2 and the remainder s that you get when you divide b by 2 and then compute $r +_2 s$.

Note that if f is a homomorphism, then the formula $f(a *_1 b) = f(a) *_2 f(b)$ extends automatically to bigger formulas like $f(a *_1 b *_1 c) = f(a) *_2 f(b) *_2 f(c)$.

There is a homomorphism from $\mathbb{Z}$ to $\mathbb{Z}_{10}$ that maps 17 to 7, 1023 to 3, -14 to 6, and more generally maps each nonnegative integer to its last digit (and maps each negative integer to 9 minus its last digit). That is, this homomorphism sends every n to $n\%10$. The homomorphism property is related to the fact that if you know the last digit of the positive integer a and the last digit of the positive integer b, then you can deduce the last digit of the positive integer $a + b$.

This is why, even though Doerr and Levasseur (and many other authors) say that (for instance) 17 is not an element of $\mathbb{Z}_{10}$ (and that only integers between 0 and 9 inclusive belong to $\mathbb{Z}_{10}$), other authors say that in the context of $\mathbb{Z}_{10}$, "17" should be construed as an alias for the number 7, and more generally every ordinary integer n should be construed as an alias for the element $f(n)$ where f is the homomorphism from $\mathbb{Z}$ to $\mathbb{Z}_{10}$.

A historically important homomorphism is the function f from $\mathbb{Z}$ to $\mathbb{Z}_9$ that sends every n to $n\%9$. It is easy to compute $n\%9$ for any positive integer n: just add the digits of n, obtaining a new number n', and then add the digits of n', obtaining a new number n'', and so on, until you arrive at a single-digit number s; if s is 9, then $n\%9 = 0$, otherwise $n\%9 = s$. For instance, with $n = 1234567$, we get $n' = 1 + 2 + 3 + 4 + 5 + 6 + 7 = 28$ and $n'' = 2 + 8 = 10$ and $n''' = 1 + 0 = 1$, so $f(n) = n\%9 = 1$. This homomorphism is at the heart of the method of casting out nines, one of the earliest attempts at error-checking. The idea is that if you added two big integers a and b and got c, then $f(c)$ should be $f(a) +_9 f(b)$ in $\mathbb{Z}_9$. Turning this around, if $f(a) +_9 f(b)$ isn't $f(c)$, then we must have made a mistake when we computed c. This method doesn't tell you where you made the mistake – only that you made one. But for many purposes that's the right first step.

I'll conclude with two examples of homomorphisms related to Exercise 11.7.8(a) from the textbook. The operation $\text{abs}(\cdot)$ from $[\mathbb{R}^*, \times]$ to $[\mathbb{R}^+, \times]$ that sends x to $|x|$ is a homomorphism because $\text{abs}(\text{x} \times \text{y}) = \text{abs}(\text{x}) \times \text{abs}(\text{y})$ (that is, $|xy| = |x||y|$) for all nonzero x, y. Likewise, the operation $\text{sign}(\cdot)$ from $[\mathbb{R}^*, \times]$ to $[\{1, -1\}, \times]$ that sends x to $+1$ if x is positive and -1 if x is negative is a homomorphism because $\text{sign}(\text{xy}) = \text{sign}(\text{x})\text{sign}(\text{y})$ for all nonzero x, y.