The order sequence of a finite group

The “order sequence of a finite group” (not a standard term, but one we’ll use in this class) is the sequence whose terms are the respective orders of all the elements of the group, arranged in increasing order.

In $\mathbb{Z}_3$ the element 0 has order 1, the element 1 has order 3, and the element 2 has order 3, so the order sequence of this group is 1,3,3.

In $\mathbb{Z}_4$ the element 0 has order 1, the element 1 has order 4, the element 2 has order 2, and the element 3 has order 4, so the order sequence of this group is 1,2,4,4. (Note that I have arranged the numbers 1,2,4,2 in increasing order.)

Theorem: If $G_1$ and $G_2$ are finite groups and $f$ is an isomorphism between them, with $g \in G_1$ and $f(g) \in G_2$, the order of $g$ in $G_1$ equals the order of $f(g)$ in $G_2$.

Consequently:

Theorem: If two groups are isomorphic, they have the same order sequence.

The theorem is a handy tool for proving that two particular groups are not isomorphic. Consider the group $U_5$ (the set $\{1,2,3,4\}$ with mod-5 multiplication). Its order sequence is 1,2,4,4, which suggests that it might be isomorphic to $\mathbb{Z}_4$. In fact, any isomorphism $f$ from $\mathbb{Z}_4$ to $U_5$ must map 0 (the only element of order 1 in $\mathbb{Z}_4$) to 1 (the only element of order 1 in $U_5$) and must map 2 (the only element of order 2 in $\mathbb{Z}_4$) to 4 (the only element of order 2 in $U_5$). There are only two bijections $f$ from $\mathbb{Z}_4$ to $U_4$ satisfying $f(0) = 1$ and $f(2) = 4$, so these are the only two candidate isomorphisms (and both candidates turn out to be true isomorphisms).