Vectors and linear (in)dependence

Vector space basics: A key example of a vector space is $\mathbb{R}^2$, in which the vectors are written as ordered pairs (x, y); the sum of two vectors is given by

$$(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2)$$

and the product of a scalar (i.e., number) and a vector is given by

$$c(x, y) = (cx, cy).$$

We often write (1,0) as $\mathbf{i}$ and (0,1) as $\mathbf{j}$, so that for instance $(3, 5) = (3, 0) + (0, 5) = 3(1, 0) + 5(0, 1) = 3\mathbf{i} + 5\mathbf{j}$. We represent (x, y) by an arrow with its tail at the origin and its head at the point written as (x, y) in Cartesian coordinates.

Vector spaces $\mathbb{R}^3, \mathbb{R}^4, \ldots$ are defined similarly (even though they're harder to picture). In $\mathbb{R}^3$, we write $\mathbf{i} = (1, 0, 0)$, $\mathbf{j} = (0, 1, 0)$, $\mathbf{k} = (0, 0, 1)$. We have $(1, 2, 4) + (3, 2, 7) = (1 + 3, 2 + 2, 4 + 7) = (4, 4, 11)$ and $5(1, 2, 4) = (5 \times 1, 5 \times 2, 5 \times 4) = (5, 10, 20)$.

Question: Can $\mathbf{k}$ be written as a linear combination of $\mathbf{i}$ and $\mathbf{j}$? That is, do there exist numbers a, b such that

$$(0, 0, 1) = a(1, 0, 0) + b(0, 1, 0)?$$

That is, do there exist numbers a, b such that $(0, 0, 1) = a(1, 0, 0)+b(0, 1, 0) = (a, 0, 0) + (0, b, 0) = (a, b, 0)$?

Answer: **No.** The vector equation $(0, 0, 1) = (a, b, 0)$ translates into the three-equation system $0 = a$, $0 = b$, $1 = 0$ which has no solution. So $\mathbf{k}$ cannot be written as a linear combination of $\mathbf{i}$ and $\mathbf{j}$.

More generally, a vector space is a set that comes equipped with two binary operations, called vector addition and scalar multiplication; the sum of two vectors is a vector, and the product of a scalar and a vector is a vector. These operations are required to satisfy various axioms, such as

$$\begin{aligned}
\mathbf{v} + \mathbf{w} &= \mathbf{w} + \mathbf{v} \\
(\mathbf{u} + \mathbf{v}) + \mathbf{w} &= \mathbf{u} + (\mathbf{v} + \mathbf{w}) \\
a(\mathbf{v} + \mathbf{w}) &= a\mathbf{v} + a\mathbf{w} \\
(a + b)\mathbf{v} &= a\mathbf{v} + b\mathbf{v} \\
1\mathbf{v} &= \mathbf{v} \\
a(b\mathbf{v}) &= (ab)\mathbf{v}
\end{aligned}$$

There exists a vector $\mathbf{0}$ (the "zero vector") such that $\mathbf{v} + \mathbf{0} = \mathbf{0} + \mathbf{v} = \mathbf{v}$ for all $\mathbf{v}$. For all $\mathbf{v}$ there exists $\mathbf{w}$ (written "$-\mathbf{v}$") such that $\mathbf{v} + \mathbf{w} = \mathbf{0}$.

For instance, let's prove that for all $\mathbf{v}$, $0\mathbf{v} = \mathbf{0}$. We have $\mathbf{0} + 0\mathbf{v} = 0\mathbf{v} = (0+0)\mathbf{v} = 0\mathbf{v} + 0\mathbf{v}$. Since $\mathbf{0} + 0\mathbf{v} = 0\mathbf{v} + 0\mathbf{v}$, we can apply the cancellation rule (a consequence of the existence of additive inverses) to deduce $\mathbf{0} = 0\mathbf{v}$, as claimed.

We can also prove that $-\mathbf{v}$ (the additive inverse of $\mathbf{v}$) is equal to $(-1)\mathbf{v}$ (the product of the scalar -1 and the element $\mathbf{v}$). Note that

$$\begin{aligned} \mathbf{v} + (-\mathbf{v}) &= \mathbf{0} \text{ (by the definition of additive inverses)} \\ &= 0\mathbf{v} \\ &= (1 + (-1))\mathbf{v} \\ &= 1\mathbf{v} + (-1)\mathbf{v} \\ &= \mathbf{v} + (-1)\mathbf{v} \end{aligned}$$

so $\mathbf{v} + (-\mathbf{v}) = \mathbf{v} + (-1)\mathbf{v}$; applying the cancellation law, we get $-\mathbf{v} = (-1)\mathbf{v}$.

We define subtraction of vectors as follows:

$$\mathbf{u} - \mathbf{v} = \mathbf{u} + (-\mathbf{v}) = \mathbf{u} + (-1)\mathbf{v}$$

Linear dependence: Linear dependence is easier to describe than its opposite (linear independence). Examples of linear dependence of vectors are

$$\mathbf{u} + \mathbf{v} = \mathbf{w}$$

and

$$\mathbf{u} + \mathbf{v} = \mathbf{w} + \mathbf{x}.$$

In the former case, we would say that the set of vectors $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is linearly dependent; in the latter case, we would say that the set of vectors $\{\mathbf{u}, \mathbf{v}, \mathbf{w}, \mathbf{x}\}$ is linearly dependent. In each case, we can write one of the vectors as a linear combination of the others; for instance, in the case of $\mathbf{u} + \mathbf{v} = \mathbf{w} + \mathbf{x}$, we can write $\mathbf{u} = \mathbf{w} + \mathbf{x} - \mathbf{v}$ or $\mathbf{v} = \mathbf{w} + \mathbf{x} - \mathbf{u}$ or $\mathbf{w} = \mathbf{u} + \mathbf{v} - \mathbf{x}$ or $\mathbf{x} = \mathbf{u} + \mathbf{v} - \mathbf{w}$. A uniform way to write linear dependencies is to move all the terms to one side of the equation, leaving $\mathbf{0}$ on the other side; for instance, $\mathbf{u} + \mathbf{v} = \mathbf{w} + \mathbf{x}$ becomes $\mathbf{u} + \mathbf{v} - \mathbf{w} - \mathbf{x} = \mathbf{0}$, or

$$(1)\mathbf{u} + (1)\mathbf{v} + (-1)\mathbf{w} + (-1)\mathbf{x} = \mathbf{0}.$$

More generally, any linear dependence between four vectors $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$, and $\mathbf{x}$ can be written in the form

$$(a)\mathbf{u} + (b)\mathbf{v} + (c)\mathbf{w} + (d)\mathbf{x} = \mathbf{0}$$

where the scalars a, b, c, and d are not all zero.

Example: If

$$(2)\mathbf{u} + (-3)\mathbf{v} + (-4)\mathbf{w} + (5)\mathbf{x} = \mathbf{0}$$

then each of the four vectors can be written as a linear combination of the other three: $\mathbf{u} = (3/2)\mathbf{v} + (4/2)\mathbf{w} + (-5/2)\mathbf{x}$, etc. (Of course it'd be fine to write 2 instead of 4/2.)

We say that a set of vectors $\{\mathbf{u}, \mathbf{v}, \ldots\}$ is linearly dependent if there exist coefficients $a, b, \ldots$, not all equal to 0, such that $a\mathbf{u} + b\mathbf{v} + \ldots$ is the zero vector $\mathbf{0}$. Confusing special case: Under this definition, if $\mathbf{v} = \mathbf{0}$, the set $\{\mathbf{v}\}$ is linearly dependent; otherwise, the set $\{\mathbf{v}\}$ is linearly independent.

We say that a set of vectors $\{\mathbf{u}, \mathbf{v}, \ldots\}$ is linearly independent if it is NOT linearly dependent. That is, a set of vectors $\{\mathbf{u}, \mathbf{v}, \ldots\}$ is linearly independent if the only choice of coefficients $a, b, \ldots$ that make the equation $a\mathbf{u} + b\mathbf{v} + \ldots = \mathbf{0}$ true is $a = 0$, $b = 0$, $\ldots$