Math 475, Problem Set #12
(due 4/27/06)

A. Chapter 8, problem 12, parts (b) and (d).

B. Find a formula for $s(n, n-2)$, valid for all $n \geq 3$.

C. Fix $n \geq 6$.

   (a) In how many ways can you partition a set with n (distinguishable) elements into 3 distinguishable boxes, so that none of the boxes contains fewer than 2 elements?

   (b) In how many ways can you partition a set with n (distinguishable) elements into into 3 *indistinguishable* boxes, so that none of the boxes contains fewer than 2 elements?

D. For each $k \geq 0$, let $\sigma_k(x) = \sum_{n=k}^{\infty} S(n,k)x^n$, the generating function for the kth column in the table of Stirling numbers of the second kind. Thus for instance $\sigma_0(x)$ is the power series $1 + 0x + 0x^2 + 0x^3 + \ldots$, also known as the constant 1.

   (a) Show that $\sigma_1(x) = x/(1-x)$.

   (b) Show that $\sigma_2(x) = x^2/(1-x)(1-2x)$.

   (c) Give a general formula (valid for all $k \geq 1$) expressing $\sigma_k(x)$ as a rational function of x. (Hint: Multiply the recurrence relation $S(n,k) = kS(n-1,k) + S(n-1,k-1)$ by x^n and sum over all $n \geq k$; then express the resulting equation in terms of σ_k and σ_{k-1}. This lets you express σ_k in terms of σ_{k-1}.)

   (d) Use part (c) in the special case $k = 3$ to find a formula for $S(p, 3)$. (Hint: Do a partial fraction decomposition of $(\sigma_3(x))/x^3$ rather than $\sigma_3(x)$.)

E. (a) Find a linear recurrence relation satisfied by the sequence of numbers $S(1,3)$, $S(2,3)$, $S(3,3)$, ... (Hint: Use the generating function $(\sigma_3(x))/x^3$ you computed above, expressed as a rational function of x, and derive the recurrence relation from the form of the denominator.)

(b) Use this to compute $S(8,3)$ from earlier terms.

(c) Compare this with the value of $S(8,3)$ obtained by using the recurrence relation $S(p,k) = S(p-1,k-1) + kS(p-1,k)$.