Math 475, Problem Set #12: Answers

A. *Chapter 8, problem 12, parts (b) and (d).*

   (b) $S^{\#}(n,2) = 2^n - 2$, since, from among the 2^n ways of putting n elements into 2 distinguishable boxes, exactly 2 of them result in one of the boxes being empty. Hence $S(n,2) = \frac{1}{2!}S^{\#}(n,2) = \frac{1}{2}(2^n - 2) = 2^{n-1} - 1$.

   Alternative solution: Let $1, \ldots, n$ denote the objects that are being put into the boxes. One we choose to put 1 into a particular box (and for purposes of counting we can ignore the fact that we have two choices since the two boxes are indistinguishable), we have a two-way choice for where to put 2, a two-way choice for where to put 3, etc.; in each case, we can either put the new number in the same box as 1 or in the other box. This would give a total of 2^{n-1} possibilities (since there are $n-1$ elements to be assigned to the boxes after 1 has been put in a box), except for the fact that we are required to leave no box empty. This will happen if all $n-1$ elements are assigned to be in the same box as 1. Hence the number of permitted assignments is not 2^{n-1} but $2^{n-1} - 1$.

   (d) When we divide a set of size n into $n-2$ non-empty subsets, we must either (a) have a set of size 3 and all other sets of size 1, or else (b) have two sets of size 2 and all other sets of size 1. Case (a): There are $\binom{n}{3}$ ways to choose the three special elements that belong to the set of size 3. Case (b): There are $\binom{n}{4}$ ways to choose the four special elements that belong to the two sets of size 2, and then there are 3 ways to divide those four elements into two sets of size 2. So there are $\binom{n}{3} + 3\binom{n}{4}$ ways to divide a set of size n into $n-2$ non-empty subsets,

B. *Find a formula for $s(n, n-2)$, valid for all $n \geq 3$.*

   When we divide a set of size n into $n-2$ non-empty circular permutations (let's call them "circles" for short), we must either (a) have a circles of size 3 and all other circles of size 1, or else (b) have two circles

of size 2 and all other circles of size 1. Case (a): There are $\binom{n}{3}$ ways to choose the three special elements that belong to the circle of size 3, and 2 ways to arrange them in a circle. Case (b): There are $\binom{n}{4}$ ways to choose the four special elements that belong to the two circles of size 2, and then there are 3 ways to divide those four elements into two circles of size 2. (Given two elements, there is only one way to arrange them in a circle of size 2.) So there are $2\binom{n}{3} + 3\binom{n}{4}$ ways to divide a set of size n into $n - 2$ circles.

C. Fix $n \geq 6$.

(a) *In how many ways can you partition a set with n (distinguishable) elements into 3 distinguishable boxes, so that none of the boxes contains fewer than 2 elements?*

Call the boxes "red", "white", and "blue". Let S be the set of all ways to of assigning the elements to boxes, let A be the set of all ways of assigning elements to boxes so that there are fewer than 2 boxes in the red box, let B be the set of all ways of assigning elements to boxes so that there are fewer than 2 boxes in the white box, and let C be the set of all ways of assigning elements to boxes so that there are fewer than 2 boxes in the blue box. We want to count the number of elements of $S - (A \cap B \cap C)$. We use the principle of inclusion-exclusion. $|S| = 3^n$, since each of the n elements can be assigned to any of the 3 boxes. $|A| = 2^n + n2^{n-1}$, since we can either assign all n elements to the white and blues boxes or assign one of the n elements to the red box and assign the remaining $n - 1$ elements to the white and blues boxes. Ditto for $|B|$ and $|C|$. $|A \cap B| = 1^n + n1^{n-1} + n1^{n-1} + n(n-1)1^{n-2}$, where the four terms correspond to the ways of having no elements in the red and white boxes, one element in the red box and no elements in the white box, no elements in the red box and one element in the white box, or one element in the red box and one (different) element in the white box. Ditto for $|A \cap C|$ and $|B \cap C|$. Lastly, $|A \cap B \cap C| = 0$, since (by the pigeonhole principle) there's no way to assign $n \geq 4$ elements to three boxes without having some box contain more than one element. So $|S - (A \cap B \cap C)| = 3^n - 3(2^n + n2^{n-1}) + 3(n^2 + n + 1)$.

(As a way of checking this, we can substitute $n = 6$, obtaining the answer 90. We can compute this a different way: It's just like the problem of assigning six different students to three different classrooms. The number of such assignments is the multinomial coefficients $\binom{6}{2,2,2} = \frac{6!}{2!2!2!} = 90$.)

(b) *In how many ways can you partition a set with n (distinguishable) elements into into 3 indistinguishable boxes, so that none of the boxes contains fewer than 2 elements?*

Let a be the answer to part (a), and b be the answer to part (b). Each assignment of the elements to 3 indistinguishable boxes gives rise to $3! = 6$ different assignments of the elements to 3 distinguishable boxes, since there are 3! ways to paint the three boxes red, white and blue. Therefore $a = 6b$. Hence $b = a/6 = (3^n - 3(2^n + n2^{n-1}) + 3(n^2 + n + 1))/6$.

D. *For each $k \geq 0$, let $\sigma_k(x) = \sum_{n=k}^{\infty} S(n,k)x^n$, the generating function for the kth column in the table of Stirling numbers of the second kind. Thus for instance $\sigma_0(x)$ is the power series $1 + 0x + 0x^2 + 0x^3 + \ldots$, also known as the constant 1.*

(a) *Show that $\sigma_1(x) = x/(1-x)$.*

We showed above that $S(n,1) = 1$ for all $n \geq 1$, so $\sigma_1(x) = x + x^2 + x^3 + \ldots = x/(1-x)$.

(b) *Show that $\sigma_2(x) = x^2/(1-x)(1-2x)$.*

We showed above that $S(n,2) = 2^{n-1} - 1$ for all $n \geq 2$, so $\sigma_2(x) = x^2 + 3x^3 + 7x^4 + 15x^5 + \ldots$, whence $(1-2x)\sigma_2(x) = x^2 + (3-2)x^3 + (7-6)x^4 + (15-14)x^5 + \ldots = x^2 + x^3 + x^4 + x^5 + \ldots = x^2/(1-x)$, so $\sigma_2(x) = x^2/(1-x)(1-2x)$.

(c) *Give a general formula (valid for all $k \geq 1$) expressing $\sigma_k(x)$ as a rational function of x. (Hint: Multiply the recurrence relation $S(n,k) = kS(n-1,k) + S(n-1,k-1)$ by x^n and sum over all $n \geq k$; then express the resulting equation in terms of σ_k and σ_{k-1}. This lets you express σ_k in terms of σ_{k-1}.)*

Following the hint, we get

$$\sum_{n \geq k} S(n,k)x^n = k \sum_{n \geq k} S(n-1,k)x^n + \sum_{n \geq k} S(n-1, k-1)x^n.$$

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The left-hand side is just $\sigma_k(x)$. The first term on the right-hand side can be re-written as $k\sum_{n\geq k+1} S(n-1,k)x^n$ (since $S(k-1,k)=0$) and then re-indexed as $k\sum_{n\geq k} S(n,k)x^{n+1} = kx\sigma_k(x)$. The second term on the right-hand side can be re-indexed as $\sum_{n\geq k-1} S(n,k-1)x^{n+1} = x\sigma_{k-1}(x)$. Thus we have

$$\sigma_k(x) = kx\sigma_k(x) + x\sigma_{k-1}(x).$$

Re-arranging, we get $(1-kx)\sigma_k(x) = x\sigma_{k-1}(x)$, which gives $\sigma_k(x) = \sigma_{k-1}(x)\frac{x}{1-kx}$. Hence, starting from $\sigma_0(x) = 1$, we get $\sigma_1(x) = \sigma_0(x)\frac{x}{1-x} = \frac{x}{1-x}$, $\sigma_2(x) = \sigma_1(x)\frac{x}{1-2x} = \frac{x}{1-x}\frac{x}{1-2x} = \frac{x^2}{(1-x)(1-2x)}$, $\sigma_3(x) = \sigma_2(x)\frac{x}{1-3x} = \frac{x}{1-x}\frac{x}{1-2x}\frac{x}{1-3x} = \frac{x^3}{(1-x)(1-2x)(1-3x)}$, etc.; the general formula is

$$\sigma_k(x) = \frac{x^k}{(1-x)(1-2x)\cdots(1-kx)}.$$

(Note that the formula $S(n,k) = kS(n-1,k) + S(n-1,k-1)$ is indeed valid for $n = k$: it just says $1 = 0+1$. Brualdi only proves this for $k \leq n-1$, and not for the case where k and n are equal. It's a good thing that the formula holds for $n = k$, since this equation got used in the preceding proof when we multiplied the equation by x^k before adding it to infinitely many other equations of the same sort!)

(d) *Use part (c) in the special case $k = 3$ to find a formula for $S(p,3)$. (Hint: Do a partial fraction decomposition of $(\sigma_3(x))/x^3$ rather than $\sigma_3(x)$.)*

We can't apply partial fractions right away, because the degree of the numerator is not less than the degree of the denominator. But for all $n \geq 3$, the coefficient of x^n in the formal power series expansion of $x^3/(1-x)(1-2x)(1-3x)$ is equal to the coefficient of x^{n-3} in the formal power series expansion of $1/(1-x)(1-2x)(1-3x)$, and this rational function of x can be decomposed by partial fractions: $1/(1-x)(1-2x)(1-3x) = A/(1-x) + B/(1-2x) + C/(1-3x)$. Multiplying both sides by $(1-x)(1-2x)(1-3x)$ and equating the the quadratic polynomials that occur on both sides, we get three linear equations, which we solve to

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obtain $A = 1/2, B = -4, C = 9/2$. So the coefficient of x^n in $1/(1-x)(1-2x)(1-3x)$ is $(1/2)(1)^n + (-4)(2)^n + (-9/2)(3)^n$, and $S(n,3) =$ the coefficient of x^n in $x^3/(1-x)(1-2x)(1-3x) =$ the coefficient of x^{n-3} in $1/(1-x)(1-2x)(1-3x) = (1/2)(1)^{n-3} + (-4)(2)^{n-3} + (9/2)(3)^{n-3} = 1/2 + (-4)/82^n + (9/2)/27(3)^n = 1/2 - (1/2)2^n + (1/6)3^n$. Note that this agrees with the formula we derived in class using inclusion-exclusion.

E. (a) *Find a linear recurrence relation satisfied by the sequence of numbers $S(1,3)$, $S(2,3)$, $S(3,3)$, ... (Hint: Use the generating function $(\sigma_3(x))/x^3$ you computed above, expressed as a rational function of x, and derive the recurrence relation from the form of the denominator.)*

Since the denominator of $1/(1-x)(1-2x)(1-3x)$ is the degree-3 polynomial $(1-x)(1-2x)(1-3x)$, and the numerator is the constant 1 (a polynomial of degree less than 3), the coefficients in the formal power series expansion of $1/(1-x)(1-2x)(1-3x)$ satisfy the recurrence whose characteristic polynomial is $(x-1)(x-2)(x-3) = x^3 - 6x^2 + 11x^2 - 6$; that is, $S(n,3) - 6S(n-1,3) + 11S(n-2,3) - 6S(n-3,3) = 0$ as long as $n > 3$. (Recall that S(0,3) is the value that doesn't fit the same pattern as the subsequent terms, so we must avoid the situation where n, $n-1$, $n-2$, or $n-3$ vanishes; that is, we need $n > 3$.) Thus we obtain the recurrence relation $S(n,3) = 6S(n-1,3) - 11S(n-2,3) + 6S(n-3,3)$, valid for all $n > 3$.

(b) *Use this to compute $S(8,3)$ from earlier terms.*

Using the initial conditions $S(1,3) = S(2,3) = 0$ and $S(3,3) = 1$, we successively derive $S(4,3) = 6(1) - 11(0) + 6(0) = 6$, $S(5,3) = 6(6) - 11(1) + 6(0) = 25$, $S(6,3) = 6(25) - 11(6) + 6(1) = 90$, $S(7,3) = 6(90) - 11(25) + 6(6) = 301$, and $S(8,3) = 6(301) - 11(90) + 6(25) = 966$.

(c) *Compare this with the value of $S(8,3)$ obtained by using the recurrence relation $S(p,k) = S(p-1,k-1) + kS(p-1,k)$.*

Using values in the table in Brualdi, we derive $S(7,3) = S(6,2) + 3S(6,3) = 31 + 3(90) = 301$ and $S(8,3) = S(7,2) + 3S(7,3) = 63 + 3(301) = 966$.