Math 475, Problem Set #13: Answers

A. *A baton is divided into five cylindrical bands of equal length, as shown (crudely) below.*

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*In how many different ways can the five bands be colored if $n$ colors are available, and unlimited repetition of the colors is allowed? (Two colorings count as the same if one of them can be converted into the other by turn the baton around.)*

We use Burnside's lemma. The group $G$ of symmetries of the baton has two elements. The identity element fixes all $n^5$ colorings. Next we count the colorings that are fixed by the other element of $G$ (which switches band 1 with band 5 and band 2 with band 4, leaving band 3 in place). If a coloring remains unchanged after a 180 rotation, the first and fifth bands must have the same color, and the second and fourth bands must have the same color. By the Multiplication Principle, the number of such colorings is $n^3$, where we assign one color to the first and fifth bands, a second color (which may be the same as the first color) to the second and fourth bands, and a third color (which may be the same as one or both of the first two colors) to the middle band. Hence the number of orbits is $\frac{1}{2}(n^5 + n^3)$.

*Alternative solution:* We need to count the number of equivalence clases (or orbits) of colorings under these symmetries. An orbit will contain either one coloring that remains unchanged after a 180 degree rotation or two different colorings that turn into each other after a 180 degree rotation. Since there are $n^5$ colorings all together, and $n^3$ of them are in orbits of size 1 (see part (a)), the remaining $n^5 - n^3$ are in orbits of size 2. Hence there are $n^3$ orbits of size 1 and $(n^5 - n^3)/2$ orbits of size 2, so that the total number of orbits is $n^3 + (n^5 - n^3)/2 = (n^5 + n^3)/2$.

B. *A circular necklace contains nine beads. Each bead is featureless, so that the necklace has no front or back, and we may flip the necklace over.*

The symmetry group of the necklace is the dihedral group discussed in the middle of page 570, in the case $n = 9$. We think of the beads as being at the vertices of a regular 9-gon. The group $G$ consists of 9 rotations and 9 reflections, where each reflection-line passes through a vertex of the 9-gon and the midpoint of the edge opposite that vertex.

(a) *How many different necklaces can be constructed if an unlimited number of red and white beads are available?*

If $g$ is the identity element, there are $2^9$ colorings that are fixed under the action $g$.

If $g$ is a reflection, there are $2^5$ colorings that are fixed under the action of $g$. (E.g., the flip that switches bead 1 with bead 9, bead 2 with bead 8, bead 3 with bead 7, and bead 4 with bead 6 while leaving bead 5 in place will not affect the appearance of the necklace as long as 1 and 9 have the same color, 2 and 8 have the same color, 3 and 7 have the same color, 4 and 6 have the same color, and 5 has whatever color one likes, for a total of $2 \times 2 \times 2 \times 2 \times 2$ colorings.)

If $g$ is rotation by 120 or 240 degrees, there are $2^3$ colorings that are fixed under the action of $g$, since beads 1, 4, and 7 must have the same color, beads 2, 5, and 8 must have the same color, and beads 3, 6, and 9 must have the same color.

If $g$ is any rotation that isn't a multiple of 120 degrees, there are only $2^1$ colorings that are fixed under the action of $g$, since all the beads must have the same color.

Hence the number of orbits is $\frac{1}{18}(2^9 + 9 \times 2^5 + 2 \times 2^3 + 6 \times 2^1) = 46$.

(b) *How many different necklaces can be constructed from three white beads and six red beads?*

Call a coloring "permissible" if it has three white and six red beads.

If $g$ is the identity element, there are $\binom{9}{3} = 84$ colorings that are fixed under the action $g$.

If $g$ is a reflection, there are 4 colorings that are fixed under the action of $g$. E.g., consider the specific flip discussed in part (a). Since there are an odd number of white beads, bead 5 must be white. Since there are three white beads all told, either beads 1 and 9 are white, beads 2 and 8 are white, beads 3 and 7 are white, or beads 4 and 6 are white.

If $g$ is rotation by 120 or 240 degrees, there are 3 colorings that are fixed under the action of $g$: either beads 1, 4, and 7 are white (and the rest are red), beads 2, 5, and 8 are white (and the rest are red), or beads 3, 6, and 9 are white (and the rest are red).

If $g$ is any rotation that isn't a multiple of 120 degrees, there are only no colorings that are fixed under the action of $g$, since all the vertices must have the same color.

Hence the number of orbits is $\frac{1}{18}(84 + 9 \times 4 + 2 \times 3 + 6 \times 0) = 7$. (They are ...WWWRRRRRR..., ...WWRWRRRRR..., ...WWRRWRRRR..., ...WWRRRWRRR..., ...WRWRWRRRR..., ...WRWRRWRRR..., and ...WRRWRRWRR...)

C. *Let $S$ be the set of ways of assigning 12 identical balls to 3 distinguishable boxes, in such a way that no box is empty, and let $G$ be the group of operations on $S$ that permute the boxes. (If it helps, you can imagine that the boxes are different colors. For instance, if $s \in S$ is the assignment that puts 5 balls in the red box, 4 balls in the white box, and 3 balls in the blue box, and $g \in G$ is the element of $G$ that switches the red box and the white box, then $gs$ or $g(s)$ is the assignment that puts 4 balls in the red box, 5 balls in the white box, and 3 balls in the blue box.)*

(a) *How many objects does $S$ contain?*

If we treat the boxes as distinguishable, then an assignment of the 12 indistinguishable balls to the 3 boxes that leaves no box empty corresponds to a triple $(a, b, c)$ of positive integers, where $a$, $b$, and $c$ are the number of balls assigned to the first, second, and third boxes, respectively, with $a + b + c = 12$. The number of such triples is $\binom{11}{2}$ or 55.

(b) *How many permutations does $G$ contain?*

$G$ has six elements: an identity element, 3 elements that swap two of the boxes while leaving one box alone, and 2 elements that exchange the boxes cyclically.

(c) *How many orbits does $S$ have under the action of $G$?*

The identity operation fixes all 55 ways to assign the indistinguishable balls to the three boxes.

Consider a swap, say the swap that exchanges box 1 and box 2. The assignments of the balls to the boxes that are fixed under the swap are $(5, 5, 2)$, $(4, 4, 4)$, $(3, 3, 6)$, $(2, 2, 8)$, and $(1, 1, 10)$. That is, there are 5 such.

Consider a cycle, say the permutation that sends box 1 to box 2, box 2 to box 3, and box 3 to box 1. The only assignment that is fixed under the cycle is $(4, 4, 4)$.

Hence the number of orbits is $\frac{1}{6}(55 + 3 \times 5 + 2 \times 1) = 12$.

(d) *Compare your answer to (c) with the value of $p_3(12)$, the number of partitions of 12 into 3 positive integers. (You can compute $p_3(12)$ however you like).*

The recurrence relation $p_k(n) = p_{k-1}(n-1)+p_k(n-k)$ specializes to $p_3(n) = p_2(n-1)+p_3(n-3)$, so we get $p_3(12) = p_2(11)+p_3(9) = p_2(11) + p_2(8) + p_3(6) = p_2(11) + p_2(8) + p_2(5) + p_3(3)$. Using the formula $p_2(n) = lfloorn/2\rfloor$ and the fact that $p_3(3) = 1$, this becomes $\lfloor 11/2\rfloor + \lfloor 8/2\rfloor + \lfloor 5/2\rfloor + 1 = 5 + 4 + 2 + 1 = 12$.

*(If it's unclear what I'm asking for, request clarification!)*

D. *Assume that an upside down I looks exactly the same as a right-side up I. How many rotationally distinct ways to decorate the faces of a cube with the letter I? (Note that, although there are two ways to color each face, this is NOT the same as asking how many rotationally distinct ways there are to color the faces of a cube with two colors!)*

Each face can be decorated with the letter $I$ in 2 ways, so there are $2^6$ decorations that are fixed under the action of the trivial symmetry.

There are NO decorations that are fixed under a 90 degree rotation about the axis joining the midpoints of two opposite faces. E.g., consider the rotation about the vertical axis. The top face and bottom face get rotated by 90 degrees, so the I looks different after the rotation gets performed.

There are $2^2 = 4$ decorations that are fixed under a 120 rotation about the axis joining two diametrically opposite corners. That's because, once you choose how to decorate the top and bottom faces, the rotation-symmetry forces the decorations of the other four faces.

There are $2^4 = 16$ decorations that are fixed under a 180 rotation about the axis joining the midpoints of two opposite faces. E.g., consider the rotation about the vertical axis. The top face and bottom face can be decorated however one likes; the front face and right face can be decorated however one likes; and the decorations of the other four faces are forced.

Lastly, there are $2^3 = 8$ decorations that are fixed under a 180 rotation about the axis joining the midpoints of two opposite edges.

Hence the number of orbits is $\frac{1}{24}(1\times 64+6\times 0+8\times 4+3\times 16+6\times 8) = 8$.

E. *In how many rotationally distinct ways can the vertices of a cube be labeled if each vertex is labeled with either a 0 or a 1?*

Each vertex can be labeled in 2 ways, so there are $2^8 = 256$ decorations that are fixed under the action of the trivial symmetry.

There are $2^2 = 4$ decorations that are fixed under a 90 degree rotation about the axis joining the midpoints of two opposite faces. E.g., consider the rotation about the vertical axis. The top four vertices must get the same color, and the bottom four vertices must get the same color.

There are $2^4 = 16$ decorations that are fixed under a 120 rotation about the axis joining two diametrically opposite corners. (Such a rotation splits up the 8 vertices as $1 + 3 + 3 + 1$.)

There are $2^4 = 16$ decorations that are fixed under a 180 rotation about the axis joining the midpoints of two opposite faces. (Such a rotation splits up the 8 vertices as $2 + 2 + 2 + 2$.)

Lastly, there are $2^4 = 16$ decorations that are fixed under a 180 rotation about the axis joining the midpoints of two opposite edges. (Such a rotation splits up the 8 vertices as $2 + 2 + 2 + 2$.)

Hence the number of orbits is $\frac{1}{24}(1\times 256+6\times 4+8\times 16+3\times 16+6\times 16) = 23$.