Math 475, Problem Set #4: Solutions

A. *Chapter 3, problem 30.*

If the person sitting to the right of the dog is a man, the people must alternate in gender thereafter, and there are 5! ways to assign the men to the five seats available for men and 5! ways to assign the women to the five seats available for women, for a total of (5!)(5!) ways. The same is true if the person sitting to the right of the dog is a woman. So the grand total number of ways is (5!)(5!) + (5!)(5!).

Alternate solution: First seat the 5 men: there are 5!/5 ways to do this (we divide by 5 since the places at the table are indistinguishable). Next we seat the 5 women at the 5 spaces between the men: there are 5! ways to do this (we don't divide by 5 because once the men are seated, the different places for the women to sit *are* distinguishable from one another). Lastly, we sit the dog in any of the 10 spaces between the men and women: there are 10 ways to do this. Hence the total number of seating arrangements is (5!/5)(5!)(10).

B. (a) *Given a row of 25 seats, in how many ways can one choose four non-overlapping blocks, each consisting of four consecutive seats?*

Represent each block of four consecutive seats by a B and every other (empty) seat by an E. There will be $25 - 4 \cdot 4 = 9$ empty seats, so we are effectively counting arrangements of 4 B's and 9 E's, of which there are $\binom{13}{4}$.

Alternate solution: For $1 \leq i \leq 3$, let x_i be the number of empty seats between the $i-1$st block and the ith block, and let x_0 (resp. x_4) be the number of empty seats to the left (resp. right) of the first (resp. last) block. Then the different ways to choose the blocks correspond to the non-negative integer solutions to $x_0 + x_1 + x_2 + x_3 + x_4 = 9$, and there are $\binom{13}{4}$ such solutions.

(b) *What if we require that there be at least one empty seat between any two blocks?*

This is effectively the same as making the blocks of size 5 and adding an extra seat at the end of the row. (That is, if seats i

through $i+3$ are in the block, we include seat $i+4$ as an honorary member of the block.) Represent each block of size 5 by a B and each seat not part of such a block by an E. We are counting arrangements of 4 B's and $26 - 4 \cdot 5 = 6$ E's, of which there are $\binom{10}{4}$.

Alternate solution: We can use the same method as we did in the second solution to part (a). Now we are adding the requirement that x_1, x_2, and x_3 are all 1 or greater. Define $y_i = x_i$ for $i = 0$ and $i = 4$, and $y_i = x_i - 1$ for $1 \leq i \leq 3$. Then the different ways to choose the blocks correspond to the non-negative integer solutions to $y_0 + y_1 + y_2 + y_3 + y_4 = 6$, of which there are $\binom{10}{4}$.

C. (a) *The number of linear permutations of the set* $\{1, 2, 3, 4, 5, 6\}$ *is equal to* 3! *times* 3! *times the number of linear permutations of the multiset* $\{o, o, o, e, e, e\}$*, since we can replace the o's by the three odd numbers in* 3! *different ways and the e's by the three even numbers in* 3! *different ways. Verify this numerically by computing the number of linear permutations of the set* $\{1, 2, 3, 4, 5, 6\}$ *and the number of linear permutations of the multiset* $\{o, o, o, e, e, e\}$*. (You need not list them all; just count them using formulas from the chapter.)*

The former is $6! = 720$ and the latter is $\binom{6}{3} = 20$; and indeed, $720 = 20 \times 6 \times 6$.

(b) *Is it also true that the number of circular permutations of the set* $\{1, 2, 3, 4, 5, 6\}$ *is equal to* 3! *times* 3! *times the number of circular permutations of the multiset* $\{o, o, o, e, e, e\}$*? As part of your answer, you should numerically compute the number of circular permutations of the set* $\{1, 2, 3, 4, 5, 6\}$ *and the number of circular permutations of the multiset* $\{o, o, o, e, e, e\}$*.* Briefly explain what is going on.

The multiset $\{o, o, o, e, e, e\}$ has only four circular permutations: one in which the three o's occur consecutively; one in which there are just two consecutive o's, followed by an e, followed by an o, followed by two e's; one in which there are two consecutive o's, followed by two e's, followed by an o, followed by an e's; and one in which e's and o's alternate. The number of circular permutations of $\{1, 2, 3, 4, 5, 6\}$ is $6!/6 = 120$, which is not equal to $4 \times 6 \times 6 = 144$.

What's going on? For one thing, the reasoning presupposes that if you start from some circular permutation of $\{o, o, o, e, e, e\}$, the 36 ways of replacing the o's by 1, 3, and 5 and the e's by 2, 4, and 6 give rise to 36 different circular permutations of $\{1, 2, 3, 4, 5, 6\}$. But this is not true: for instance, if you start from some circular permutation of $\{o, o, o, e, e, e\}$ in which o's and e's alternate, which we write as o, e, o, e, o, e, then one replacement gives $1, 2, 3, 4, 5, 6$ and another replacement gives $3, 4, 5, 6, 1, 2$, which are the same circular permutation of $\{1, 2, 3, 4, 5, 6\}$.

(c) Note that the number of linear permutations of the set $\{1, 2, 3, 4, 5, 6\}$ is equal to 6 times the number of circular permutations of the set $\{1, 2, 3, 4, 5, 6\}$. Is it also true that the number of linear permutations of the multiset $\{o, o, o, e, e, e\}$ is equal to 6 times the number of circular permutations of the multiset $\{o, o, o, e, e, e\}$? Briefly explain what is going on.

No, it is not: $20 \neq 6 \times 4$. Each of the six way of breaking open any circular permutation of $\{1, 2, 3, 4, 5, 6\}$ and turning it into a linear permutation (by starting at some point and reading off the entries in clockwise order) gives rise to a different linear permutation of the set, but this is not true for all circular permutations of the multiset $\{o, o, o, e, e, e\}$. In particular, it fails for o, e, o, e, o, e: we can only get two different linear permutations of the multiset, not six, by breaking open the circular permutation o, e, o, e, o, e.

D. *Chapter 3, problem 37.*

Apply Theorem 3.5.1: We have $k = 6$ different types of pastry, from which we are to choose $r = 12$ pastries. Since the pastries of each type are unlimited, we are effectively in the situation where each type of pastry has infinite repetition number, so the number of combinations is $\binom{r+k-1}{k-1} = \binom{17}{5}$.

If we are to required to take at least one pastry of each of the six kinds, then we can imagine that we choose these first, and then choose six more without any constraints. In choosing six more pastries, we are in the situation where $r = 6$ and $k = 6$, so the number of combinations is $\binom{r+k-1}{k-1} = \binom{11}{5}$.

E. *How many different poker hands are there that contain two pairs (of two different ranks) and a fifth card of a different rank than the other four cards?*

To specify a hand that contains two pairs, we must choose the two ranks that will contain the pairs (which we can do in $\binom{13}{2} = 78$ ways) and choose one of the remaining $13 - 2 = 11$ ranks for the rank of the odd card out (which we can do in 11 ways); then we must specify which $\binom{4}{2} = 6$ of the cards in the first selected rank are in the hand, which $\binom{4}{2} = 6$ of the cards in the second selected rank are in the hand, and which of the 4 cards in the remaining selected rank is the odd card out. Hence the number of two-pairs hands is $78 \times 11 \times 6 \times 6 \times 4 = 123552$.

(If you got twice this number, it's probably because you first chose the rank of one pair, and then chose the rank of the other pair. This leads to double-counting, because, for instance, a hand that contains two kings and two queens could have been gotten in two ways, by choosing two kings and then choosing two queens or by choosing two queens and then choosing two kings. Make sure you see why this double-counting problem doesn't occur when we're counting hands that contain a three-of-a-kind and a pair!)

F. *Twelve students taking a potions class are to be divided into four working-groups of three students apiece. All we care about are which students are working together and which students are not. In how many ways can the students be divided into the groups? Be sure to relate your answer to the answer to problem F from the previous assignment.*

Let x be the number of ways of dividing the students into four groups of size three, and let y be the solution to problem F on the previous assignment. There are 4! ways of matching up these four groups with the four houses at the academy, yielding a way of assigning three students to each house. Furthermore, each way of assigning the students to the houses arises exactly once in this fashion. Hence $y = (4!)x$, by the multiplication principle. Solving, we get $x = y/4! = 12!/3!3!3!3!4!$.

Note that this is just like assigning objects to boxes where the boxes are unlabeled (see Theorem 3.4.3).