Math 475, Problem Set #5: Solutions

A. *Chapter 3, problem 28. Do part (a) in two different ways: once by brute force (i.e., dynamic programming), and once by interpreting the counting of routes in terms of multiset permutations. Likewise, do part (b) in two different ways: once by dynamic programming, and once by multiset permutations (making use of Brualdi's hint as well). You may use a calculator or computer to facilitate the dynamic programming computation.*

The brute force way to do part (a):

1	9	45	165	495	1287	3003	6435	12870	24310
1	8	36	120	330	792	1716	3432	6435	11440
1	7	28	84	210	462	924	1716	3003	5005
1	6	21	56	126	252	462	792	1287	2002
1	5	15	35	70	126	210	330	495	715
1	4	10	20	35	56	84	120	165	220
1	3	6	10	15	21	28	36	45	55
1	2	3	4	5	6	7	8	9	10
1	1	1	1	1	1	1	1	1	1

The slick way to do part (a): A path consist of 9 Eastward steps and 8 Northward steps. So each path corresponds to a permutation of the multiset $\{9 \cdot E,\ 8 \cdot N\}$, of which there are

$$\binom{9+8}{8} = (17)(16)(15)(14)(13)(12)(11)(10)/(8)(7)(6)(5)(4)(3)(2)(1) = 24310.$$

The brute force way to do part (b):

1	9	45	165	495	1252	2793	5700	10910	19900
1	8	36	120	330	757	1541	2907	5210	8990
1	7	28	84	210	427	784	1366	2303	3780
1	6	21	56	126	217	357	582	937	1477
1	5	15	35	70	91	140	225	355	540
1	4	10	20	**35**	**21**	49	85	130	185
1	3	6	10	15	21	28	36	45	55
1	2	3	4	5	6	7	8	9	10
1	1	1	1	1	1	1	1	1	1

The slick way to do part (b): The paths that we *don't* want to count — those that are unusable because they use the easterly block that is submerged —- are those that go from the lower left (Home) to the **35**, go east to the **21**, and then go to the upper right (Work). The number of such paths is $\binom{4+3}{3}\binom{4+5}{5}$ (where $\binom{4+3}{3}$ counts the paths that start at the lower left and then go 4 blocks east and 3 blocks north to arrive at the **35**, and $\binom{4+5}{5}$ counts the paths that start at the **21** and then go 4 blocks east and 5 blocks north to arrive at the upper right). So the number of bad paths is $\binom{7}{3}\binom{9}{5} = (35)(126) = 4410$ and the number of good paths is $24310 - 4410 = 19900$.

B. *Chapter 3, problem 40.*

(a) We can see directly that this is $C(n, k)$. Or, we can associate each way of choosing k of the n sticks with a $(k+1)$-tuple $(x_0, x_1, \ldots, x_k)$ where x_0 is the number non-chosen stick to the left of the first chosen stick, x_k is the number non-chosen stick to the right of the last chosen stick, and every other x_i $(1 \leq i \leq k-1)$ is the number of non-chosen sticks between the ith and $i+1$st chosen sticks. This gives a $(k+1)$-tuple of non-negative integers with sum $n-k$, and every such $(k+1)$-tuple arises in this way from a unique way of choosing k of the n sticks. So the number of ways of choosing k of the n sticks equals the number of $(k+1)$-tuples $(x_0, x_1, \ldots, x_k)$ of non-negative integers with $x_0 + x_1 + \ldots + x_k = n - k$. Putting $n' = k+1$ and $r' = n-k$, and applying what we already know about counting sequences of non-negative integers with a prescribed sum, we see that the number of such $(k+1)$-tuples is $C(n' + r' - 1, n' - 1) = C(n, k)$.

(b) Associate each way of choosing k of the n sticks with a $(k+1)$-tuple $(x_0, x_1, \ldots, x_k)$ where x_0 is the number non-chosen stick to the left of the first chosen stick, x_k is the number non-chosen stick to the right of the last chosen stick, and every other x_i $(1 \leq i \leq k-1)$ is the number of non-chosen sticks between the ith and $i+1$st chosen sticks. Saying that no two of the chosen sticks can be consecutive is equivalent to saying that $x_1, x_2, \ldots, x_{k-1}$ are all positive. Defining $y_0 = x_0$ and $y_k = x_k$ and $y_i = x_i - 1$ for all $1 \leq i \leq k-1$, we obtain a $(k+1)$-tuple $(y_0, y_1, \ldots, y_k)$ of non-negative integers with sum $n - k - (k-1) = n - 2k + 1$, and every such $(k+1)$-tuple arises in this way from a unique way of choosing k of the n sticks with no two of the chosen sticks being consecutive. Putting $n' = k+1$ and $r' = n-2k+1$, and applying what we already know about counting sequences of non-negative integers with a prescribed sum, we see that the number of such $(k+1)$-tuples is $C(n'+r'-1, n'-1) = C(n-k+1, k)$. (Check: If $n = 2k-1$, there is a unique way of choosing k of the n sticks so that no two are consecutive — namely, you must take every other stick, starting with the leftmost — and sure enough, in this case we get $C(n-k+1, k) = C(k, k) = 1$.)

(c) Associate each way of choosing k of the n sticks with a $(k+1)$-tuple $(x_0, x_1, \ldots, x_k)$ where x_0 is the number non-chosen stick to the left of the first chosen stick, x_k is the number non-chosen stick to the right of the last chosen stick, and every other x_i $(1 \leq i \leq k-1)$ is the number of non-chosen sticks between the ith and $i+1$st chosen sticks. Saying that there must be at least l non-chosen sticks between any two of the chosen sticks is equivalent to saying that $x_1, x_2, \ldots, x_{k-1}$ are all greater than or equal to l. Defining $y_0 = x_0$ and $y_k = x_k$ and $y_i = x_i - l$ for all $1 \leq i \leq k-1$, we obtain a $(k+1)$-tuple $(y_0, y_1, \ldots, y_k)$ of non-negative integers with sum $n-k-(k-1)l = n-k+l-kl$, and every such $(k+1)$-tuple arises in this way from a unique way of choosing k of the n sticks with no two of the chosen sticks being consecutive. Putting $n' = k+1$ and $r' = n-k+l-kl$, and applying what we already know about counting sequences of non-negative integers with a prescribed sum, we see that the number of such $(k+1)$-tuples is $C(n'+r'-1, n'-1) = C(n+l-kl, k)$. (Check: If $n = (k-1)l+k$, there is a unique way of choosing k of the n sticks so that no two are consecutive — namely, you must take the 1st stick, then the $l+2$nd, then the $2l+3$rd, etc., up through the $(k-1)l+k$th — and sure enough, in this case we get $C(n+l-kl, k) = C(k, k) = 1$.)

C. (a) *Chapter 3, problem 48. Do this problem directly in terms of multiset permutations. (Hint: Look at the special case $m = n = 2$. What reversible operation might you perform on a string of 3 A's and 2 B's that would turn it into a string of 2 A's and at most 2 B's?)*

Given a permutation of $m+1$ A's and n B's (of length $m+n+1$), we can delete the final A and all the B's that come after it. What results is a sequence that contains exactly m A's and up to n B's. Going in reverse, given a sequence that contains exactly m A's and up to n B's, we can tack an A on the end, along with as many B's afterward as are needed to make the total number of B's equal to n. So there is a one-to-one correspondence between the permutations of $m+1$ A's and n B's (of which there are $\binom{m+n+1}{m+1}$) and the permutations of m A's and at most n B's.

(b) *Use the addition principle (just once) to show that*

$$p(m,m)+p(m+1,m)+p(m+2,m)+\ldots+p(m+n,m) = p(m+n+1,m+1),$$

where $p(\cdot,\cdot)$ is as section 5.1.

Every path from $(0,0)$ to $(m+n+1,m+1)$ eventually takes a diagonal-downward step from the $j=m$ column to the $j=m+1$ column. Let this step more specifically be $(i,m) \to (i+1,m+1)$. This is the last diagonal step, and it is followed by vertical steps.

Now let S be the set of paths from $(0,0)$ to $(m+n+1,m+1)$, and let S_i $(m \le i \le m+n)$ be the set of paths from $(0,0)$ to $(m+n+1,m+1)$ whose last diagonal step goes from (i,m) to $(i+1,m+1)$. Since the sets S_i are disjoint and have S as their union, we have

$$|S| = |S_m| + |S_{m+1}| + \ldots + |S_{m+n}|.$$

But note that $|S_i|$ is just $p(i,m)$, and $|S|$ itself is $p(m+n+1,m+1)$, so the claim is proved.

(c) *Explain the relationship between parts (a) and (b) of this problem.*

The truth of (a) implies the truth of (b) and vice versa, because there is a one-to-one correspondence between permutations of m A's and k B's (on the one hand) and paths from $(0,0)$ to $(m,m+k)$ (on the other), by way of the construction discussed in class. In fact, the proof we used in part (b) is just the pictoral version of the proof we used in part (a), because removing the last diagonal edge of a path (and all the vertical edges that come after it) is tantamount to removing the last A in a multiset permutation (and all the B's that come after it).

D. *Chapter 3, problem 49. Find and fix Brualdi's mistake. (Hint: Look at the special case $m=n=1$. What reversible operation might you* *perform on a string of 2 A's and 2 B's that would turn it into a string of at most 1 A and at most 1 B? If you're stuck for ideas, take another look at part (a) of the preceding problem!)*

The correct formula is $\binom{m+n+2}{m+1} - 1$.

Given any sequence of $m+1$ A's and $n+1$ B's other than $A \cdots AB \cdots B$, the first B precedes the last A, so if we remove the last A and all the B's that come after it, and we remove the first B and all the A's that come before it, we obtain a string containing at most m A's and at most n B's. To reverse this operation, stick an A at the end, stick a B at the beginning, and then stick extra B's at the end and extra A's at the beginning so as to make the total number of A's equal to $m+1$ and the total number of B's equal to $n+1$. So there is one-to-one correspondence between the permutations of $m+1$ A's and $n+1$ B's (of which there are $\binom{m+n+2}{m+1}$) and the permutations of at most m A's and at most n B's.

Brualdi probably subtracted 2 because he was thinking that the "empty permutation" (with 0 A's and 0 B's) wouldn't strike students as being a valid permutation. Yet it really should count, if only so as to make Theorem 3.4.3 valid for all choices of $n_1, \ldots, n_k$, some of which may be zero (so why not permit them *all* to be zero)?

E. *Let $f(n)$ be the nth Fibonacci number, so that $f(1) = 1$, $f(2) = 2$, and $f(n) = f(n-1) + f(n-2)$ for all $n \geq 3$. Prove by induction that the sum $f(1) + f(2) + \ldots + f(n)$ is equal to $f(n+2) - 2$, for all $n \geq 1$.*

Let $g(n) = f(1) + f(2) + f(3) + \ldots + f(n)$.

Claim: $g(n) = f(n+2) - 2$. Proof by induction: We have $g(1) = 1 = 3 - 1 = f(3) - 2$. Suppose the claim is true for $n - 1$, so that $g(n-1) = f((n-1)+2) - 2 = f(n+1) - 2$. Then $g(n) = f(1) + f(2) + \ldots + f(n) = (f(1) + \ldots + f(n-1)) + f(n) = g(n-1) + f(n) = (f(n+1) - 2) + f(n) = (f(n) + f(n+1) - 2 = f(n+2) - 2$, so the claim holds for n as well. Hence by induction the claim is true for all n.