Math 491, Problem Set #12: Solutions

1. *Define the diagonal of a two-variable generating function*

$$F(x,y) = \sum_{m,n} a_{m,n} x^m y^n$$

as the generating function

$$D(t) = \sum_n a_{n,n} t^n.$$

It is a theorem (which we will not have time to prove) that the diagonal of any two-variable rational generating function is an algebraic generating function. Verify this claim in the particular case $F(x,y) = 1/(1-x-y) = \sum_{m,n} \frac{(m+n)!}{m!n!} x^m y^n$ by expressing the diagonal $D(t)$ as an algebraic function. Give as good a justification of your formula as you can.

From our work on Catalan numbers, we know that $\frac{1-\sqrt{1-4t}}{2t} = \sum_{n=0}^{\infty} \left(\binom{2n}{n}/(n+1) \right) t^n$; if we multiply by t and differentiate, we cancel the $n+1$ in the denominator, obtaining $(1-4t)^{-1/2} = \sum_{n=0}^{\infty} \binom{2n}{n} t^n$.

2. *Call a sequence of $+1$'s 0's, and -1's* favorable *if every partial sum is non-negative and the total sum is 0. Let $f(n)$ be the number of favorable sequences of length n. Express the generating function $\sum_n f(n)x^n$ as an algebraic function of x.*

Let $F(x)$ be the generating function for all favorable sequences, and $P(x)$ be the generating function for just the primitive ones, where a favorable sequence is called primitive iff it cannot be written as a concatenation of two or more non-empty favorable sequences. We have $F(x) = 1 + P(x)F(x)$ on general principles. Furthermore, $P(x) = x + x^2F(x)$, since a primitive favorable sequence is either a sequence of length 1 whose sole term is 0 or else a $+1$ followed by a favorable sequence followed by a -1. Hence

$$F(x) = 1 + (x + x^2F(x))F(x),$$

which gives the quadratic equation $x^2F(x)^2 + (x-1)F(x) + 1 = 0$. Solving, we get

$$F(x) = \frac{1 - x - \sqrt{1 - 2x - 3x^2}}{2x^2}.$$