Math 491, Problem Set #13: Solutions

1. *An "augmented Aztec diamond of order n" is a subset of the square grid formed by symmetrically stacking rectangles of height 1 and respective widths $2, 4, 6, \ldots, 2n-2, 2n, 2n, 2n, 2n-2, \ldots, 6, 4, 2$ (note that there are 3 rectangles of width $2n$ being stacked). Express the generating function $\sum_{n\geq 0} a_n t^n$ as an algebraic function of t, where a_n denotes the number of domino-tilings of the augmented Aztec diamond of order n, and we put $a_0 = 1$. (Note that the augmented Aztec diamond of order 1 is just a rectangle of height 3 and width 2.)*

   Divide the region into unit squares, each of which has two vertical edges of length 1; put a dot at the midpoint of a vertical edge if the sum of the x and y coordinates of the edge has the same parity as the sum of the x and y coordinates of the two points where the horizontal symmetry axis crosses the boundary of the region. As in the examples studied in class, the dominos that have two dots on their boundaries (rather than a single dot in their interior) give rise to paths that join up dots on the boundary of the region. Since there are only two dots on the boundary of the region (one at the left and one at the right), the domino tilings of the augmented Aztec diamond of order n correspond to paths from $(0,0)$ to $(2n,0)$ in which the legal steps are $(+1,+1)$, $(+1,-1)$, and $(+2,0)$.

   Call a path with steps of this kind "high" if it never goes below the line $y = 0$, and "low" if it never goes above the line $y = 1$. (Note that the trivial path, with no arcs, is both high and low.) Also call a path "primitive" if it is non-trivial and it intersects the line $y = 0$ only at its left and right endpoints. (Note that a single horizontal arc of length 2 counts as primitive, high, and low.) Let a_n^{high} be the number of high paths joining $(0,0)$ and $(2n,0)$ (which is also the number of low paths joining these points), let a_n^{prim} be the number of primitive paths joining $(0,0)$ and $(2n,0)$, and let $a_n^{high,prim}$ be the number of paths that are both high and primitive. Put $A(x) = \sum_{n\geq 1} a_n x^n$, and define $A^{high}(x)$, $A^{prim}(x)$, and $A^{high,prim}(x)$ analogously. Since every path is a concatenation of primitive paths in a unique way, we have

$$A(x) = 1/(1 - A^{prim}(x)).$$

On the other hand, a primitive path is either a high primitive path or a low primitive path, and these two cases only overlap when the primitive path is a rightward arc of length 2, so that

$$A^{prim}(x) = 2A^{high,prim}(x) - x.$$

A high primitive path either is a single rightward arc or else consists of an upward diagonal followed by a high path followed by a downward diagonal, so that

$$A^{high,prim}(x) = x + xA^{high}(x).$$

Finally, a high path is a concatenation of high primitive paths in a unique way, so

$$A^{high}(x) = 1/(1 - A^{high,prim}).$$

These last two equations refer only to the two unknowns A^{high} and $A^{high,prim}$, so we can solve for both, obtaining in particular $A^{high,prim}(x) = (1 + x - \sqrt{1 - 6x + x^2})/2$. Plugging this into the second equation, we get $A^{prim}(x) = 1 - \sqrt{1 - 6x + x^2}$, and plugging this in turn into the first equation, we get $A(x) = 1/\sqrt{1 - 6x + x^2}$.

2. *Use the exchange principle for 2-routings to find a compact formula for the number of lozenge-tilings of the semiregular hexagon with side-lengths $a, b, 2, a, b, 2$.*

   Assume the sides of length 2 are vertical. As in class, draw a dot at the midpoint of every vertical edge, so that every horizontal tile has a dot in its middle and every non-horizontal tile has two dots on its boundary, which we can join up to form paths. There are four dots on the boundary of the hexagon: two on the left (call them s_1 and s_2, from top to bottom) and two on the right (call them t_1 and t_2, from top to bottom).

   Is it easy to see that tilings of such a region are equivalent to 2-routings of a DAG that join s_1 with t_1 and s_2 with t_2. The number of paths joining s_1 with t_1 is $(a + b)!/a!b!$, as is the number of paths joining s_2 with t_2. On the other hand, the number of paths joining s_1 with t_2 is $(a + b)!/(a - 1)!(b + 1)!$ while the number of paths joining s_2 with t_1 is $(a + b)!/(a + 1)!(b - 1)!$. Since every way of joing s_1 with t_2 and s_2

with t_1 leads to an intersection, the exchange principle applies, and we conclude that the number of 2-routings is

$$\left(\frac{(a+b)!}{a!b!}\right)^2 - \left(\frac{(a+b)!}{(a-1)!(b+1)!}\right)\left(\frac{(a+b)!}{(a+1)!(b-1)!}\right),$$

which simplifies to

$$\frac{(a+b)!(a+b+1)!}{a!(a+1)!b!(b+1)!}.$$