Math 491, Problem Set #16: Solutions

1. (a) *How many lattice paths from* $(0,0)$ *to* (m,n) *remain the same when you rotate them by 180 degrees about* $(\frac{m}{2},\frac{n}{2})$*? Prove your answer.*

By symmetry, the lattice path must cross the line $x+y=(m+n)/2$ at the point $(m/2,n/2)$, which is impossible if m and n are both odd (since for every point on the lattice path, either the x-coordinate or the y-coordinate is an integer).

If m and n are both even, then $(m/2,n/2)$ is a lattice point, and we can see that every lattice path from $(0,0)$ to $(m/2,n/2)$, when rotated 180 degrees about the point $(m/2,n/2)$, yields a lattice path from $(0,0)$ to (m,n) that is invariant under 180 rotation. Since it is also clear that every invariant lattice path is of this form, the number of such paths is just $\frac{(m/2+n/2)!}{(m/2)!(n/2)!}$.

If m is even and n is odd, then $(m/2,n/2)$ is the midpoint of the segment joining $(m/2,(n-1)/2)$ and $(m/2,(n+1)/2)$, and this segment must be part of the lattice path. In particular, the lattice path must go from $(0,0)$ to $(m/2,(n-1)/2)$. In this case, the lattice paths from $(0,0)$ to (m,n) that are invariant under rotation are in bijection with the lattice paths from $(0,0)$ to $(m/2,(n-1)/2)$, and the number of paths is just $\frac{(m/2+n/2-1/2)!}{(m/2)!(n/2-1/2)!}$.

Likewise, if m is odd and n is even, the number of invariant paths is $\frac{(m/2+n/2-1/2)!}{(m/2-1/2)!(n/2)!}$.

2. (a) *How many lattice paths from* $(0,0)$ *to* (n,n) *remain the same when you flip them across the diagonal joining* $(n,0)$ *and* $(0,n)$*? Prove your answer.*

Such a path must cross the line $x+y=n$ at some point (i,j). If we flip the path from $(0,0)$ to (i,j) across the diagonal, we get a lattice path from $(0,0)$ to (n,n) of the specified kind, and every such path arises in this way. Thus the paths from $(0,0)$ to (n,n) that are invariant under reflection are in bijection with lattice paths from $(0,0)$ to the line $x+y=n$, of which there are $\binom{n}{0}+\binom{n}{1}+\ldots+\binom{n}{n}=2^n$.

(b) *What is the sum of the q-weights of these lattice paths? Conjecture an answer.*

Consider a lattice path from $(0,0)$ to (i,j), with $i+j=n$. If its q-weight is q^m, then the associated path from $(0,0)$ to (n,n) (obtained by reflection) has q-weight q^{2m+j^2}. (To see this, split the area under the path into three parts: the part below and to the left of (i,j), the part above and to the right of (i,j), and the part below and to the right of (i,j); these regions have area m, m, and j^2, respectively.) Therefore, using the function $P_{m,n}(q)$ from the previous problem set, we can write the sum of the q-weights of the symmetrical lattice paths from $(0,0)$ to (n,n) as $\sum_{j=0}^{n} q^{j^2} P_{n-j,j}(q^2)$.

Here we can use Maple. First, from what we learned in class, we can write a program for $P_{m,n}$:

```
P := proc(m,n) local i,j;
        expand(simplify(mul(mul(
        (1-q^(i+j))/(1-q^(i+j-1)),
        i=1..m),j=1..n))); end;
```

Then we can write a program to q-count reflection-invariant lattice paths

```
S := proc(n) local j;
        expand(simplify(add(q^(j^2)*
        subs(q=q^2,P(n-j,j)),j=0..n)));
        end;
```

A little bit of exploration will then yield the observation that $S(n)$ divided by $S(n-1)$ equals $1+q^{2n-1}$, so that

$$S(n) = (1+q)(1+q^3)\cdots(1+q^{2n-1}).$$

In fact, we can prove this combinatorially by dividing the area under the lattice path into L-shapes with their corners at the lower right. Each L-shape, being symmetrical, contains an odd number of squares. Also, since the L-shapes fit together to form a (flipped) Young diagram, their sizes must be distinct, with the the largest possible L-shape being of size $2n-1$. Finally, note that if we take any set of odd numbers from 1 to $2n-1$, L-shapes of those sizes may be fit together to form a flipped Young diagram that is reflection-invariant and whose boundary is a reflection-invariant lattice path.

(c) *Why is there no part (b) for question 1?*

Because all of the paths have the same q-weight, namely $q^{AB/2}$, so it would have been silly to ask the question.

(Actually, the preceding paragraph *would* have been the right answer, if someone else had asked the question, or if I had asked it in class. But since I asked the question *on the homework*, I obviously didn't think it was too silly a question to ask! A psychologically accurate answer is that I originally included a part (b), then deleted it, and then decided to re-include it, but in a slightly off-beat way that would hopefully be amusing or at least provocative.)