Math 491, Problem Set #18: Solutions

1. *Use Lindstrom's lemma, the interpretation of domino tilings as routings, and a computer, in order to count the domino tilings of an 8-by-8 square. (You will receive no credit for merely giving the correct answer.)*

   Checkerboard-color the squares in the grid, so that the upper-left square is shaded. Mark the mid-point of every vertical edge that has a black square to its left or a white square to its right (or both). It's easy to check that every possible placement of a domino yields either zero or two marked points on its boundary. Hence, if one fixes a domino tiling and draws connections between all pairs of marked points that share a domino, one gets four non-intersecting left-to-right lattice paths joining the four leftmost marked points to the four rightmost marked points. Conversely, given four such lattice paths, one can construct a tiling by taking all those dominoes that cover an edge of the lattice path, along with all dominoes that are centered on those marked points that do not lie on any of the lattice paths. Hence there is a bijection between domino-tilings of the 8-by-8 grid and families of non-intersecting lattice paths joining the sources s_1, s_2, s_3, s_4 to the sinks t_1, t_2, t_3, t_4 in a trellis-like directed graph, with directed edges corresponding to the vectors $(1,1)$, $(1,-1)$, and $(2,0)$. It is easy to see that the only way to connect the s_i's and the t_j's via non-intersecting paths in this directed graph is to connect s_i to t_i for $1 \leq i \leq 4$. Hence Lindstrom's Lemma applies, and the number of families of non-intersecting lattice paths is equal to the determinant of the 4-by-4 matrix M whose i,jth entry equals the number of lattice paths from s_i to t_j.

   To determine the entries of M, we introduce new vertices in a shifted lattice that fills the holes in the lattice of marked points. (That is to say, we now associated a point with every vertical edge.) The points s_1, s_2, s_3, s_4 are the 2nd, 4th, 6th, and 8th points on the left edge (and similarly for t_1, t_2, t_3, t_4). Then the i,jth entry of M is equal to the

$2i, 2j$th entry of $AA^TAA^TAA^TAA^T$, where

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix}.$$

Using Maple, one gets

```
[ 22   68   30   48   10   12    1    1 ]
[                                       ]
[ 68  236  116  216   60   84   13   14 ]
[                                       ]
[ 30  116   62  128   41   61   11   12 ]
[                                       ]
[ 48  216  128  320  129  230   60   70 ]
[                                       ]
[ 10   60   41  129   63  128   40   48 ]
[                                       ]
[ 12   84   61  230  128  306  116  146 ]
[                                       ]
[  1   13   11   60   40  116   52   68 ]
[                                       ]
[  1   14   12   70   48  146   68   90 ]
```

Extracting the sub-matrix

```
[ 236  216   84   14 ]
[                    ]
[ 216  320  230   70 ]
[                    ]
[  84  230  306  146 ]
[                    ]
[  14   70  146   90 ]
```

and taking its determinant, one gets 12988816.

2. *Using the bijection between tilings and routings discussed in class, Lindstrom's lemma, and Dodgson condensation, prove that for all $a, b \geq 0$ and for $c = 3$, the number of ways to tile an a, b, c, a, b, c semiregular hexagon with unit rhombuses is equal to*

$$\frac{H(a+b+c)H(a)H(b)H(c)}{H(a+b)H(a+c)H(b+c)}$$

where $H(0) = H(1) = 1$ and $H(n) = 1!2!3! \cdots (n-1)!$ for $n > 1$.

I might as well prove the claim for all c (though you didn't have to). Let $T(a, b, c)$ denote the number of rhombus tilings of the a, b, c, a, b, c semiregular hexagon. It is easy to check that for all $a, b \geq 0$, $T(a, b, 0) = 1 = \frac{H(a+b+0)H(a)H(b)H(0)}{H(a+b)H(a+0)H(b+0)}$ and $T(a, b, 1) = \frac{(a+b)!}{(a)!(b)!} = \frac{H(a+b+1)/H(a+b)}{(H(a+1)/H(a))(H(b+1)/H(b)}$ $= \frac{H(a+b+1)H(a)H(b)H(1)}{H(a+b)H(a+1)H(b+1)}$. We will prove the claim for $c > 1$ using induction on c. (For the homework problem, you don't need induction; you just reduce the case $c = 3$ to the case $c = 2$ already solved in an earlier homework.)

Rhombus-tilings of the a, b, c, a, b, c semiregular hexagon correspond to routings with c sources and c sinks in a directed graph in which the number of paths from the ith source to the jth sink equals $\binom{a+b}{b-i+j}$. Therefore by Lindstrom's lemma we have $T(a, b, c) = \det M(a, b, c)$ where $M(a, b, c)$ denotes the c-by-c matrix whose i, jth entry is $\binom{a+b}{b-i+j}$. In view of the this, Dodgson condensation tells us that

$$\begin{aligned} T(a,b,c)T(a,b,c-2) &= T(a,b,c-1)^2 \\ &\quad -T(a+1,b-1,c-1)T(a-1,b+1,c-1). \end{aligned}$$

For slight notational convenience, I'll re-index this as

$$T(a,b,c+1)T(a,b,c-1) = T(a,b,c)^2 - T(a+1,b-1,c)T(a-1,b+1,c).$$

The problem now reduces to algebraically verifying that $T(a, b, c+1)$ must be given by the $H(\ )$-formula if $T(a, b, c-1)$, $T(a, b, c)$, $T(a+1, b-1, c)$ and $T(a-1, b+1, c)$ are. Equivalently, we must verify that if all five of these $T(\ )$-values are as given by the $H(\ )$-formula, then the expression

$$T(a,b,c+1)T(a,b,c-1) - T(a,b,c)^2 + T(a+1,b-1,c)T(a-1,b+1,c)$$

must vanish.

If we trust Maple, then we can prove this by noting that the final command in the string of commands

```
H := proc(n) product(k!,k=1..n-1); end;
T := proc(a,b,c) H(a+b+c)*H(a)*H(b)*H(c)
        /H(a+b)/H(a+c)/H(b+c); end;
U := T(a,b,c)*T(a-2,b,c)-T(a-1,b,c)^2
        +T(a-1,b-1,c+1)*T(a-1,b+1,c-1);
simplify(expand(U));
```

gives the output `0`. However, if you're more skeptical, here's a sketch of how you can show by hand that the expression $T(a,b,c+1)T(a,b,c-1) - T(a,b,c)^2 + T(a+1,b-1,c)T(a-1,b+1,c)$ vanishes when each $T(\ )$ is expanded using the $H(\ )$-formula. Write each of the three terms as a fraction, and in each of the terms divide the numerator by $H(a+b+c-1)H(a+b+c)H(a-1)H(a)H(b-1)H(b)H(c-1)H(c)$ and the denominator by $H(a+b)^2H(a+c-1)H(a+c)H(b+c-1)H(b+c)$, obtaining another messy expression. But we have made progress: where before we had a sum each term of which was a ratio of products each factor of which was a value of the H-function, we now have a sum each term of which is a ratio of products each factor of which is a value of the factorial function, Moreover, there are now some factors common to all three terms; removing them gives

$$\frac{(a+b+c)!(a-1)!(b-1)!(c)!}{(a+c)!(b+c)!}$$
$$-\frac{(a+b+c-1)!(a-1)!(b-1)!(c-1)!}{(a+c-1)!(b+c-1)!}$$
$$+\frac{(a+b+c-1)!(a)!(b)!(c-1)!}{(a+c)!(b+c)!}.$$

Removing common factors again gives us

$$(a+b+c-1)(c-1) - (a+c-1)(b+c-1) + (a)(b),$$

which vanishes.