Math 491, Problem Set #4: Solutions

1. (a) *Does there exist a polynomial $p(t)$ of degree 3 such that the linear operator $p(T)$ annihilates the sequence whose nth term (for $n \geq 0$) is $3^n + 2^n + 1^n$? Exhibit such a polynomial or explain why none exists.*

   Since $T - 3I$ annihilates 3^n, and $T - 2I$ annihilates 2^n, and $T - I$ annihilates 1^n, the linear operator $(T - 3I)(T - 2I)(T - I) = T^3 - 6T^2 + 11T - 6$ annihilates $3^n + 2^n + 1^n$.

   (b) *Same as (a), but with "degree 3" replaced by "degree 4".*

   Any linear operator of the form $(T - cI)(T^3 - 6T^2 + 11T - 6)$ will do; e.g., with $c = 0$, we get $T^4 - 6T^3 + 11T^2 - 6T$.

   (c) *Same as (a), but with "degree 3" replaced by "degree 2".*

   No. Suppose we have constants A, B, C such that $AT^2 + BT + CI$ annihilates the sequence whose nth term is $f(n) = 3^n + 2^n + 1^n$. That is, suppose $Af(n+2) + Bf(n+1) + Cf(n) = 0$ for all $n \geq 0$. Substituting $n = 0$, $n = 1$, and $n = 2$ into this equation we get $14A + 6B + 3C = 0$, $36A + 14B + 6C = 0$, and $98A + 36B + 14C = 0$. Since the determinant

$$\begin{vmatrix} 14 & 6 & 3 \\ 36 & 14 & 6 \\ 98 & 36 & 14 \end{vmatrix}$$

   is not equal to zero, the only solution to this linear system is $A = B = C = 0$.

2. *Let F_n be the nth Fibonacci number, as Wilf indexes them (with $F_0 = F_1 = 1$, $F_2 = 2$, etc.). Give a simple homogeneous linear recurrence relation satisfied by the sequence whose nth term is...*

   (a) nF_n*:*

   This sequence is given by a formula of the form $Anr^n + Bns^n$ (since $F_n = Ar^n + Bs^n$), where r and s are the roots of $t^2 - t - 1 = 0$. So we need a polynomial which has r as a double root and s as a double root. $(t^2 - t - 1)^2 = t^4 - 2t^3 - t^2 + 2t + 1$ will certainly do. So, writing the nth term of the given sequence as f_n, we have $f_{n+4} = 2f_{n+3} + f_{n+2} - 2f_{n+1} - f_n$.

Alternatively, we can use generating functions: If $F_0 + F_1x + F_2x^2 + F_3x^3 + \ldots = 1/(1-x-x^2)$, then, differentiating, we have $1F_1 + 2F_2x + 3F_3x^2 + \ldots = (1+2x)/(1-x-x^2)^2$, and the occurrence of $(1-x-x^2)^2 = 1-2x-x^2+2x^3+x^4$ in the denominator tells us that the sequence must satisfy the recurrence $f_{n+4} = 2f_{n+3} + f_{n+2} - 2f_{n+1} - f_n$.

(b) $1F_1 + 2F_2 + \ldots + nF_n$:

If we apply the operator $T-I$ to this sequence, we get the sequence considered in part (a). So the sequence f_n whose nth term is $1F_1 + \ldots + nF_n$ is annihilated by the operator $(T-I)(T^4 - 2T^3 - T^2 + 2T + I) = T^5 - 3T^4 + T^3 + 3T^2 - T - I$.

Alternatively, we can use generating functions, and multiply the formal power series $(1+2x)/(1-x-x^2)^2$ (considered in the previous sub-problem) by $1 + x + x^2 + \ldots = 1/(1-x)$. The coefficients of the resulting formal power series are easily seen to be partial sums of exactly the desired kind. So the new denominator is $(1-x)(1-x-x^2)^2 = 1 - 3x + x^2 + 3x^3 - x^4 - x^5$, which tells us that the sequence must satisfy the recurrence $f_{n+5} = 3f_{n+4} - f_{n+3} - 3f_{n+2} + f_{n+1} + f_n$.

(c) $nF_1 + (n-1)F_2 + \ldots + 2F_{n-1} + F_n$: This sum is the coefficient of x^n in the product of the formal power series $F_1x + F_2x^2 + \ldots + F_nx^n + \ldots$ with the formal power series $1 + 2x + 3x^2 + \ldots + nx^{n-1} + \ldots$. The former is given by a formal power series with denominator $1-x-x^2$ and the latter is given by a formal power series with denominator $(1-x)^2$; when we multiply them, we get a formal power series with denominator $(1-x-x^2)(1-x)^2 = 1 - 3x + 2x^2 + x^3 - x^4$, so the sequence satisfies the recurrence $f_{n+4} = 3f_{n+3} - 2f_{n+2} - f_{n+1} + f_n$.

(d) *F_n when n is odd, and 2^n when n is even:* We saw in class that the Fibonacci numbers satisfy the recurrence $f_{n+4} = 3f_{n+2} - f_n$. On the other hand, the powers of two satisfy the recurrence $f_{n+2} = 4f_n$. Since any multiple of $T^4 - 3T^2 + I$ annihilates the former, and any multiple of $T^2 - 4I$ annihilates the latter, an operator that annihilates both sequences (while only looking two, four, or six terms earlier) is $(T^4 - 3T^2 + I)(T^2 - 4I) = T^6 - 7T^4 + 13T^2 - 4I$. So $f_{n+6} = 7f_{n+4} - 13f_{n+2} + 4f_n$.